

2.2. Continuous Time Random Processes

S countable state space.

A continuous time random process

$$(X_t)_{t \geq 0} = (X_t : 0 \leq t < \infty)$$

$$X_t : \Omega \rightarrow S$$

Qn. How do we specify probabilities associated to $(X_t)_{t \geq 0}$?

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

Does not extend to uncountable disjoint union.

disjoint

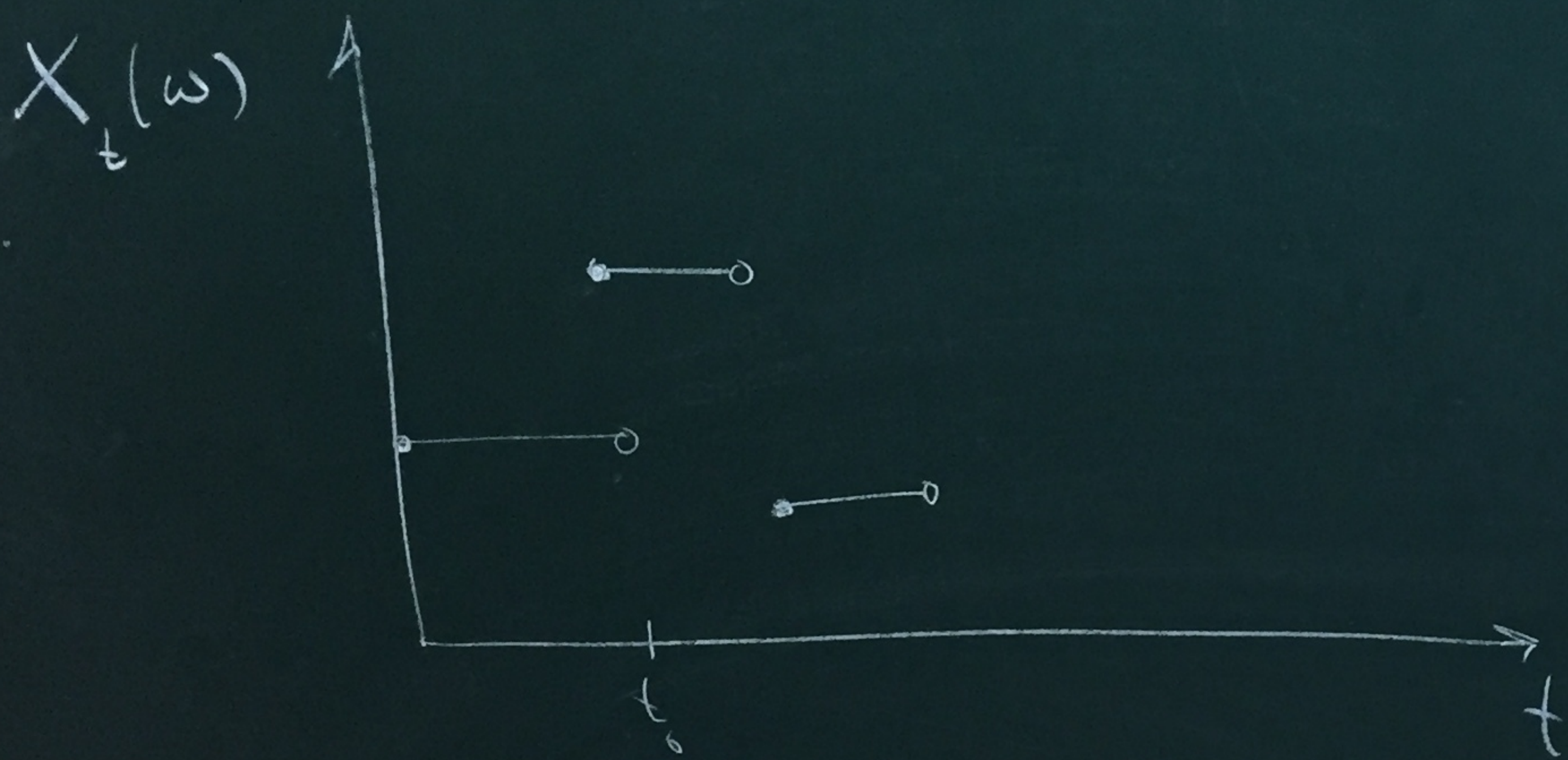
$$\mathbb{P}(X_t = i \text{ for some } t \geq 0), \quad \mathbb{P}(X_{t_0} = i), \quad \mathbb{P}(X_{t_0} = i_0, X_{t_1} = i_1, \dots, X_{t_n} = i_n)$$

$X_t(\omega)$

Assumption: Restrict to right continuous processes.

For all $\omega \in \Omega$ and $t \geq 0$, there exists $\varepsilon > 0$ such that

$$X_s(\omega) = X_t(\omega) \quad \text{for } t \leq s \leq t + \varepsilon$$



$$f: [0, \infty) \rightarrow \mathbb{Q}$$

$$q_n \downarrow 0$$

$$\lim_{h \downarrow 0} f(h) = f(0)$$

$$f(q_n) = q_n$$

2.2. Continuous Time Random Processes

Consequence (not proved) that Probability of any event of the process is determined from the finite dimensional distributions

$$P(X_{t_0} = i_0, X_{t_1} = i_1, \dots, X_{t_n} = i_n)$$

$$n \geq 0 \quad 0 \leq t_0 \leq t_1 \leq \dots \leq t_n$$

$$i_0, i_1, \dots, i_n \in S$$

Eg

$$\mathbb{P}(X_t = i \text{ for some } i) = 1 - \mathbb{P}(X_t \neq i \text{ for any } t \geq 0) \quad ?$$

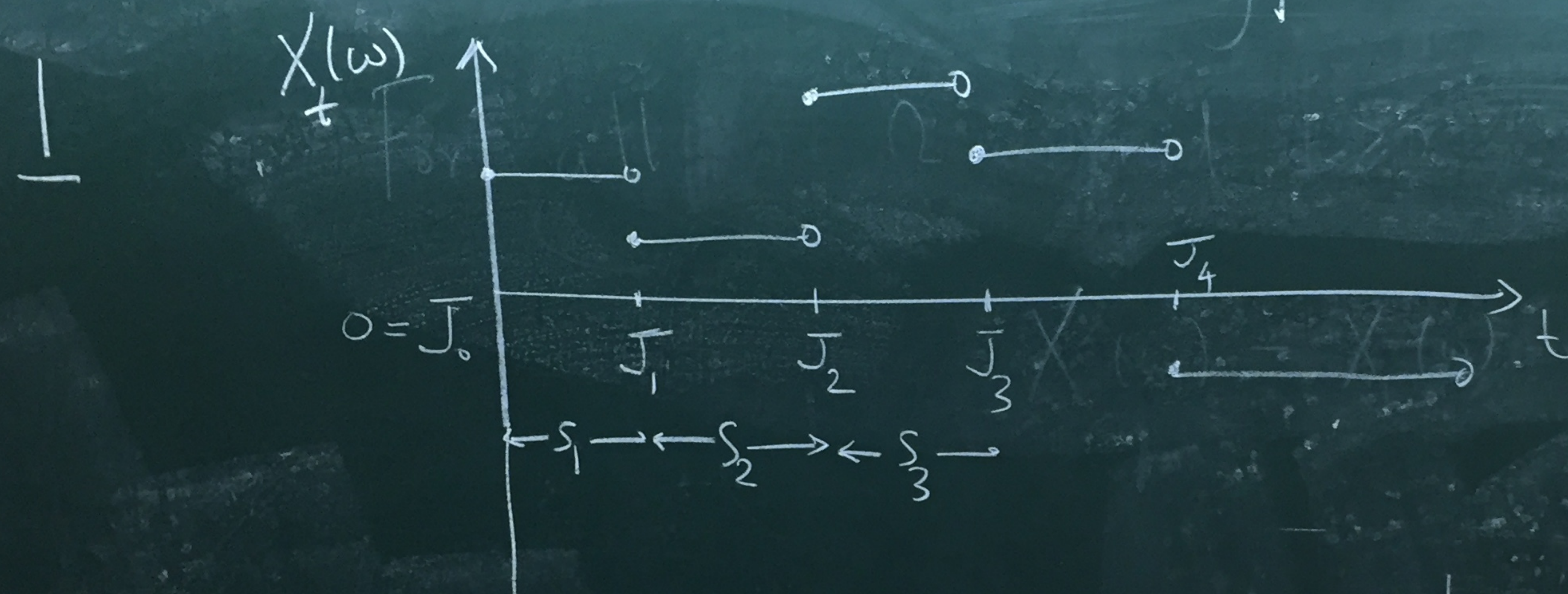
$$= 1 - \mathbb{P}(X_q \neq i \text{ for any } q \in \mathbb{Q} \cap \mathbb{R}_+)$$

$$= 1 - \lim_{n \rightarrow \infty} \mathbb{P}(X_{q_1} \neq i, X_{q_2} \neq i, \dots, X_{q_n} \neq i)$$

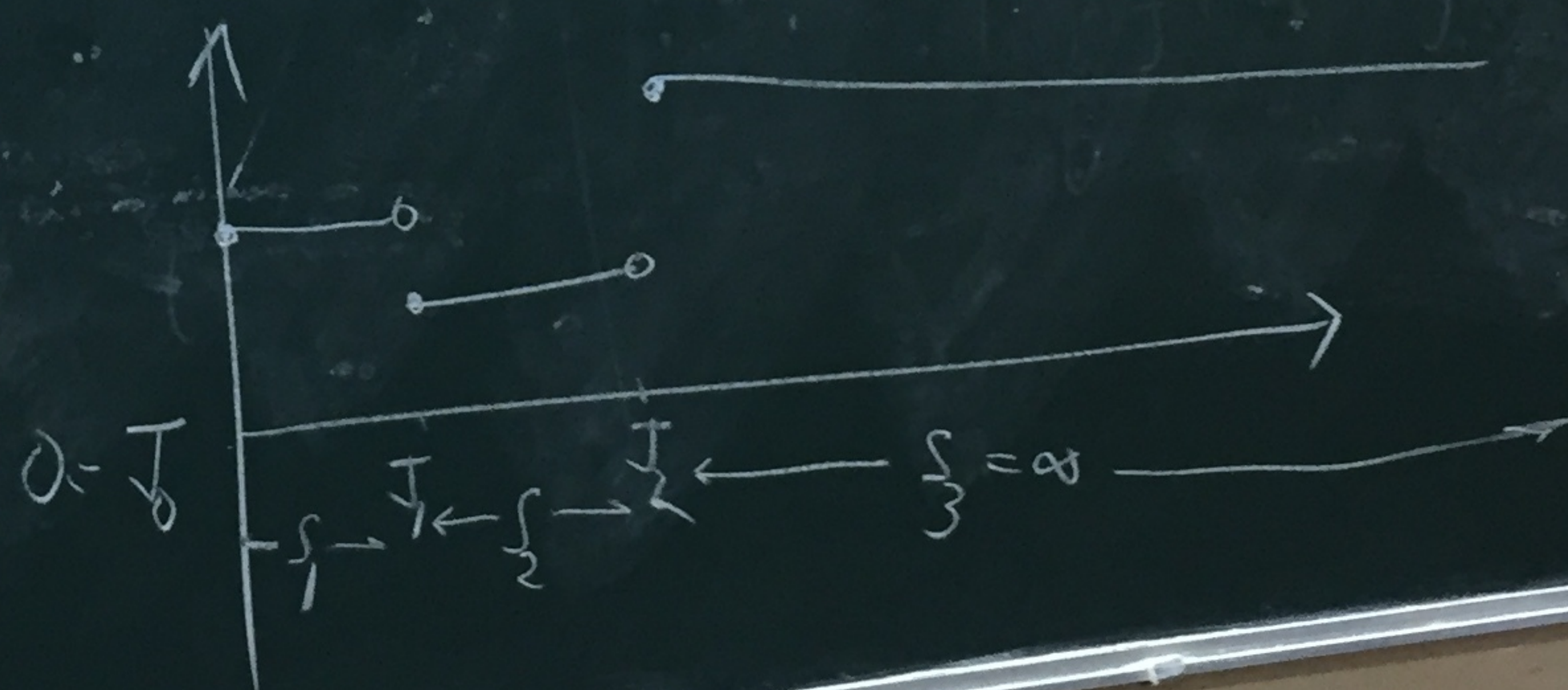
$$= 1 - \lim_{n \rightarrow \infty} \sum_{j_1, \dots, j_n \neq i} \mathbb{P}(X_{q_1} = j_1, \dots, X_{q_n} = j_n)$$

$q_1, q_2, \dots$ enumeration
of $\mathbb{Q} \cap \mathbb{R}_+$

Q_n: How do right continuous processes look like?

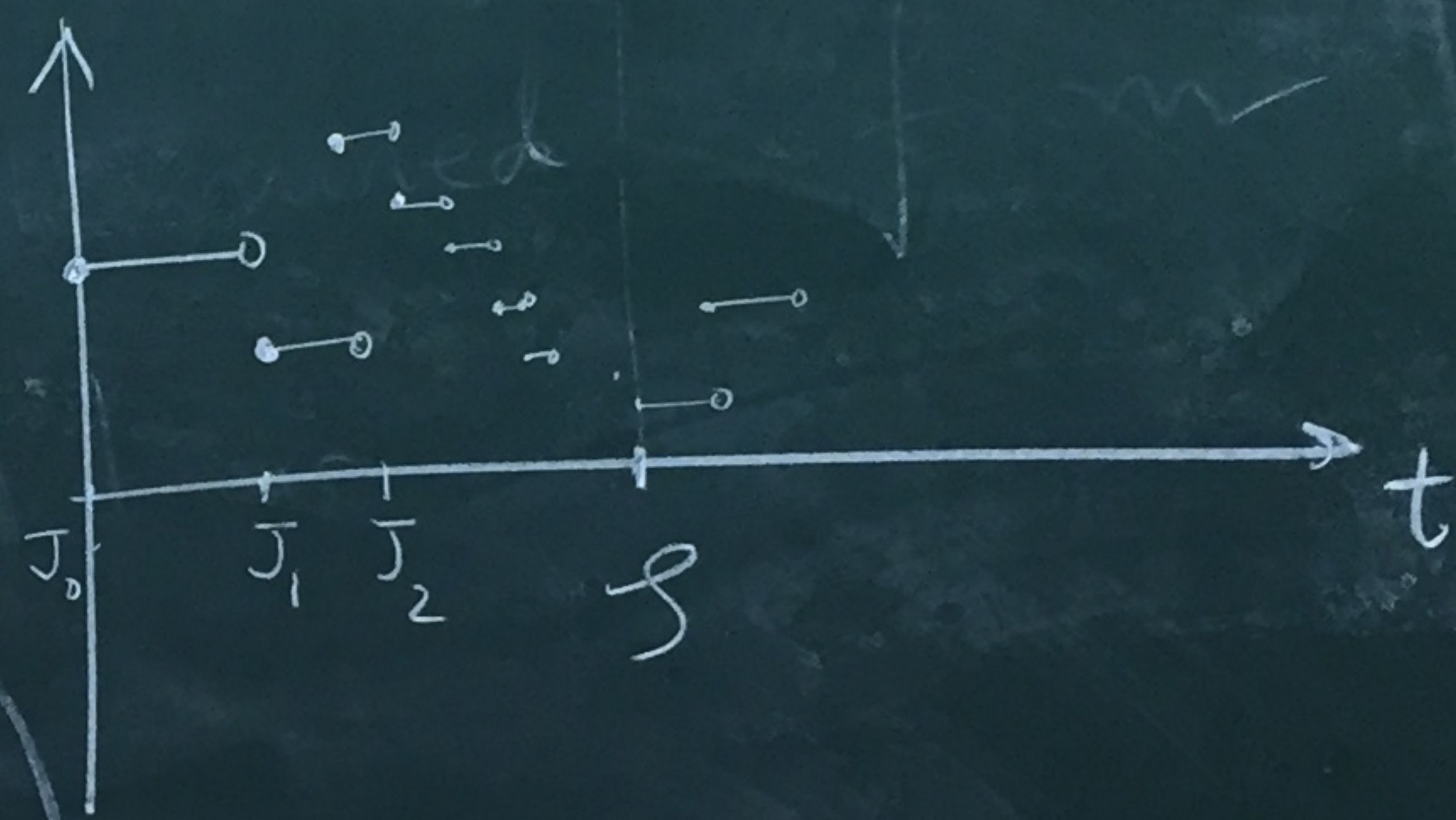


2 Makes finitely many jumps and become stuck in a state



2.2 Continuous Time Random Processes

B Infinitely many jumps in a finite interval
upto an explosion time ζ



Defn

$$J_0 = 0$$

$$J_{n+1} = \inf \{ t \geq J_n : X_t \neq X_{J_n} \}$$

$$(\inf \phi = \infty)$$

jump times
of $(X_t)_{t \geq 0}$

Think
about
it

Think
about
it

Fix t Fix $s_n \downarrow t$

$$P(X_{s_n} \neq X_t \text{ for large } n) = 0 \text{ for any } (t, s)/X?$$

For each ω , $N_t(\omega) = \sup \{d: X_s \text{ is constant in } [t, t+d]\}$

For each ω , $N_t(\omega) > 0$

$$P[\omega: N_t(\omega) > \varepsilon] \rightarrow 1 \text{ as } \varepsilon \downarrow 0$$

$$P(X > \delta) = 1$$

$$P(X > \varepsilon) \rightarrow 1 \text{ as } \varepsilon \downarrow 0$$

$$S_n = \begin{cases} T_n - T_{n-1} & \text{if } T_{n-1} < \infty \\ \infty & \text{otherwise} \end{cases}, n \geq 1$$

holding times

Remark

By right continuity, $S_n > 0$ for all n

If $T_{n+1} = \infty$ for some n then $X_\infty = X_{T_n}$, ^{undefined} otherwise

The (first) explosion time ζ is

$$\zeta = \sup_n T_n = \sum_{n=1}^{\infty} S_n$$

2.2. Continuous Time Random Processes

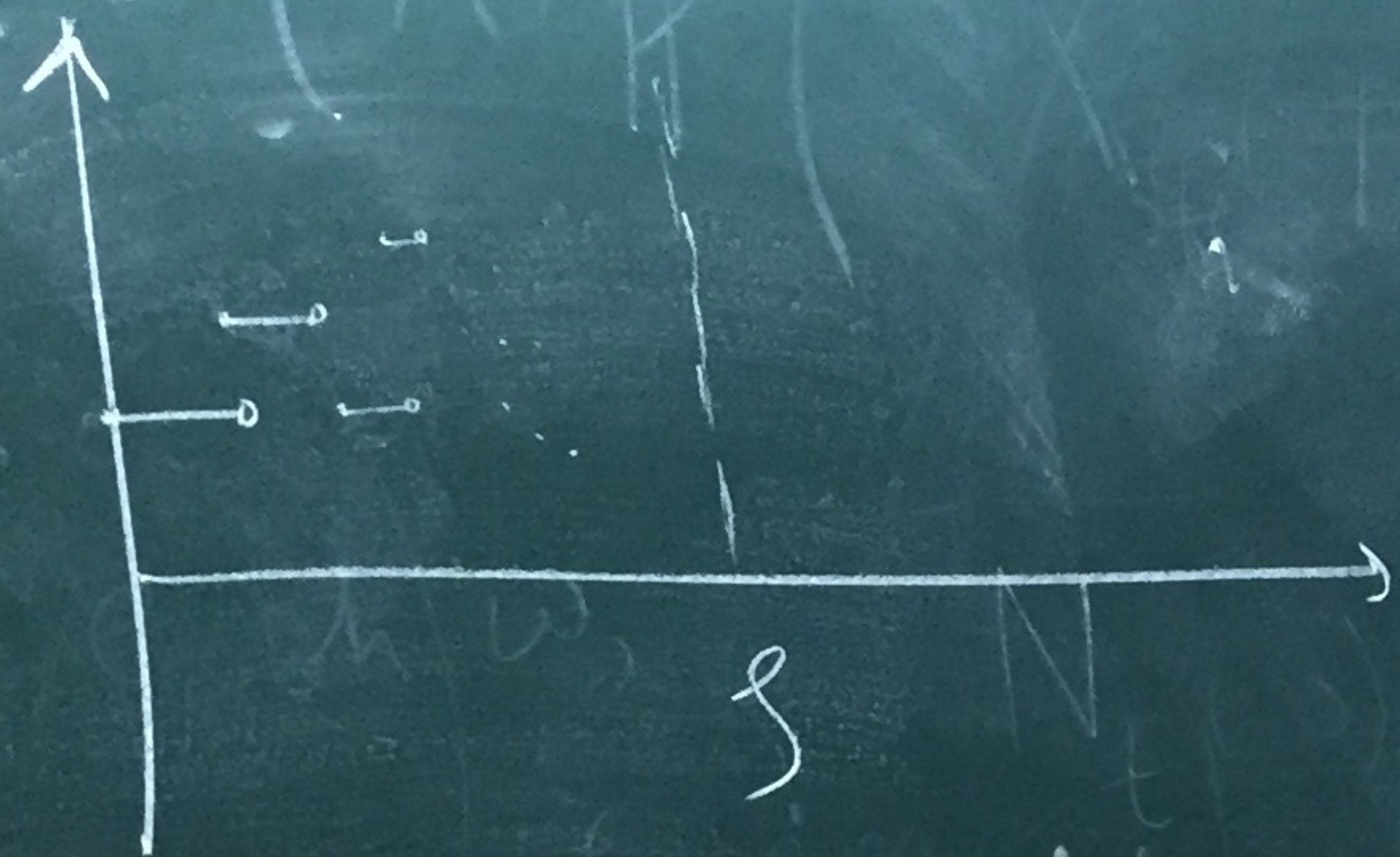
$(Y_n)_{n \geq 0}$ with $Y_n = X_{\frac{n}{j}}$ is the jump process of $(X_t)_{t \geq 0}$.

Called jump chain if we have $(X_n, n \in \mathbb{Z}_+)$ instead of $(X_t, t \geq 0)$.

Remark: We consider our process only up to explosion time ζ . Add to S a new state, ∞ , so that

$X_t = \infty$ if $t \geq \zeta$. Such a process is minimal.

Think about it



$$Y_n = X_{J_n}$$

Remark:

Can construct minimal process from $(Y_n)_{n \geq 0}$ & $(S_n)_{n \geq 1}$

$$\mathbb{P}(X_t = i) = \sum_{n=0}^{\infty} \mathbb{P}(Y_n = i \text{ and } J_n \leq t < J_{n+1})$$

$$\mathbb{P}(X_t = i \text{ for some } t) = \mathbb{P}(Y_n = i \text{ for some } n \geq 0)$$

2.3 Exponential distribution

Def. A random variable $T: \Omega \rightarrow [0, \infty)$ has exponential distribution of parameter λ ($0 < \lambda < \infty$) if $P[T > t] = e^{-\lambda t}$ for all $t \geq 0$

$T \sim \text{Exp}(\lambda)$ Density $f_T(t) = \lambda e^{-\lambda t} \mathbf{1}_{\{t \geq 0\}}$

Then $E(T) = \int_0^\infty P(T > t) dt = \frac{1}{\lambda}$

2.2. Continuous Time Random Processes

Thm A random variable $T: \Omega \rightarrow (0, \infty]$ has
an exponential distribution $\iff$
(Exs) $P[T > s+t \mid T > s] = P[T > t]$
for all $s, t \geq 0$

Remark: For continuous time Markov procen, the amount
of time to be spent at a particular state in the
future should not depend on how much time
spent in past. From above, holding times must be exponential

Think
about
in

Thm

$$P(S_1 < \infty) = 1 \quad (1)$$

$$E(S_1) = \infty$$

$$S_2, S_3, \dots = 0 \quad (2)$$

P
 T

Think $\uparrow$ Fix t

Thm

Let $S_1, S_2, \dots$ be a sequence of independent random variables

$$S_n \sim \text{Exp}(\lambda_n)$$

$$0 < \lambda_n < \infty$$

$$P(S_1 < \infty) = 1 \quad (1)$$

$$E(S_1) = \infty$$

$$S_2, S_3, \dots = 0 \quad (2)$$

$$\text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$$

$$\text{then } P\left(\sum_{n=1}^{\infty} S_n < \infty\right) = 1$$

$$\text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$$

$$\text{then } P\left(\sum_{n=1}^{\infty} S_n = \infty\right) = 1$$

$$(1) \quad E\left(\sum_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} E(S_n) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$$

$$S_n \quad P\left(\sum_{n=1}^{\infty} S_n < \infty\right) = 1$$

$$E(X_1 + \dots + X_N) = \sum_{n=1}^N E(X_n)$$

Independence & MCT $E[\dots]$

be Exponential

2.3 Monotone Convergence Theorem:

$$X_n \geq 0, \quad X_n \uparrow X, \quad (X_n(\omega) \uparrow X(\omega) \text{ for all } \omega)$$

$$\text{Then } \mathbb{E}X_n \uparrow \mathbb{E}X$$

$$\text{If } X_n \downarrow X, \quad \mathbb{E}X_1 < \infty \quad \text{then } \mathbb{E}X_n \downarrow \mathbb{E}X$$

$$(iii) \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty \quad \Rightarrow \quad \prod_{n=1}^{\infty} \left(1 + \frac{1}{\lambda_n}\right) = \infty$$

Independence
& MCT

$$\mathbb{E} \left[\exp \left(- \sum_{n=1}^{\infty} S_n \right) \right] = \prod_{n=1}^{\infty} \mathbb{E} \exp(-S_n) = \prod_{n=1}^{\infty} \left(1 + \frac{1}{\lambda_n}\right)^{-1} = 0$$

$$\text{So } \mathbb{P} \left(\sum_{n=1}^{\infty} S_n = \infty \right) = 1$$