Recall:

Thm 2.1.2 Q - matrix on S, $|S| < \infty$. $P(t) = e^{tQ}$

$t \geq 0$.

(i) $P(s+t) = P(s) P(t)$, for all s, t

(ii) $\{ P(t) : t \geq 0 \}$ is unique solution to the

$$\frac{d}{dt} P(t) = P(t) Q \qquad P(0) = I$$

(iii) .. backward

(iv)

$$\left. \frac{d^m}{dt^m} P(t) \right|_{t=0} = Q^m \qquad m \in \{0\} \cup \mathbb{N}$$

Thm 2.1.3 A matrix Q on a finite set S.

Q is a Q-matrix $\iff$ $P(t) = e^{tQ}$ is a stochastic matrix.

Proof:

$$\sum_{j \in S} q_{ij} = 0 \iff \sum_{j \in S} p_{ij}(t) = 1 \quad \forall i \in S \quad \text{— } (*)$$

$i \neq j$, $q_{ij} > 0 \implies p_{ij}(t) > 0 \quad \forall t > 0 \quad \text{— } (**)$

$\because$ $0 < t < 1$ $\exists c_1 > 0$ such that

$$-c_1 t^2 + q_{ij} t \leq p_{ij}(t) \leq q_{ij} t + c_1 t^2 \quad \text{— } (***)$$

choose $\delta > 0$ such that $p_{ij}(t) > 0$ $0 < t < \delta$ and $[P(t/n)]^n = P(t)$

$i = j \qquad -q_{ii} > 0$

$$p_{ii}(t) = 1 + q_{ii}t + \sum_{k=2}^{\infty} \frac{t^k q_{ii}^k}{k!}$$

As in (xxx) $\quad 0 < t < 1 \quad \exists\, c_2 > 0$

$$1 + q_{ii}t - c_1t^2 \le p_{ii}(t) \le 1 + q_{ii}t + c_1t^2$$

$\Rightarrow \quad \exists\, \delta > 0$ such that $p_{ii}(t) > 0 \quad 0 < t < \delta$

Use $[P(t/m)]^m = P(t)$ to conclude $p_{ii}(t) > 0 \quad \forall t > 0$

At, $t = 0 \qquad P(0) = I \qquad$ So $\Rightarrow \quad p_{ij}(0) = 0 \quad p_{ii}(0) = 1.$

So we have shown if Q is Q-matrix and $q_{ij} > 0 \;\; \forall i,j \in S \;\; i \ne j$

$\Rightarrow \quad P(t) = e^{tQ}$ is a stochastic matrix

$P(t)$

let Q - be any Q matrix.

let $\varepsilon > 0$ be given.

$$q^{\varepsilon}_{ij} = \begin{cases} \varepsilon & q_{ij} = 0 \\ q_{ij} & q_{ij} > 0 \end{cases} \quad i \neq j$$

$P(t) = e^{tQ}$

$$q^{\varepsilon}_{ii} = q_{ii} - \varepsilon \, \#\{j : q_{ij} = 0\}$$

let $Q^{\varepsilon} = [q^{\varepsilon}_{ij}]$ is - Q - matrix such that

$$q^{\varepsilon}_{ij} > 0 \quad \forall i \neq j$$

let $P^{\varepsilon}(t) = e^{tQ^{\varepsilon}}$. By earlier argument

$$p^{\varepsilon}_{ij}(t) > 0 \quad \forall i, j \in S.$$

Observe:

$$q_{ij}^{\varepsilon} \longrightarrow q_{ij} \quad \text{as } \varepsilon \to 0 \quad \forall\, i,j \in S$$

$$q_{ij}^{\varepsilon,k} \longrightarrow q_{ij}^{k} \quad \text{as } \varepsilon \to 0 \quad \forall\, k \in \mathbb{N},\ \forall\, i,j \in S$$

where

$$\left[q_{ij}^{\varepsilon,k}\right]_{i,j\in S} = \left(Q^{\varepsilon}\right)^{k}$$

So

$$p_{ij}^{\varepsilon}(t) = \sum_{k=0}^{\infty} \frac{q_{ij}^{\varepsilon,k} t^k}{k!} \xrightarrow[\text{as } \varepsilon \to 0]{(\text{Ex.})} \sum_{k=0}^{\infty} \frac{q_{ij}^{k} t^k}{k!}$$

But

$$p_{ij}(t) = \sum_{k=0}^{\infty} \frac{q_{ij}^{k} t^k}{k!} \Rightarrow \lim_{\varepsilon \to 0} p_{ij}^{\varepsilon}(t) = p_{ij}(t)$$

$$p_{ij}^{\varepsilon}(t) > 0 \ \ \forall t > 0 \Rightarrow p_{ij}(t) \geq 0 \ \ \forall t > 0$$

$P(0) = I$

& By (*) $\Rightarrow$ $P(t)$ is a stochastic matrix for all $t \geq 0$.

[Converse] let Q be such that $P(t) = e^{tQ}$ & $P(t)$ be stochastic

Thm 2.1.2 $\Rightarrow$ $P(0) = I$ & $\frac{d}{dt} P(t) = P(t) Q$.

$$\left.\frac{dP(t)}{dt}\right|_{t=0} = Q$$

$\Rightarrow$ $p_{ij}(0) = 0 \quad \forall i \neq j \quad i,j \in S$

& $\left.\frac{d}{dt} p_{ij}(t)\right|_{t=0} = q_{ij}$

& $p_{ij}(t) \geq 0 \quad \forall t > 0$

As $p_{ij}(t) \geq 0$ for $t \geq 0$ & $p_{ij}(\cdot)$ is differentiable

$p_{ij}(0) = 0$

$\Rightarrow q_{ij} \geq 0$

As $(*)$ $\Rightarrow \sum_{j \in S} q_{ij} = 0 \Rightarrow 0 \leq -q_{ii} < \infty$, since $|S| < \infty$.

Suppose P - stochastic matrix such that $P = e^{Q}$

for some matrix Q.

Fix $m \geq 1$, let $\{X_n^{(m)}\}_{n \geq 0}$ be a Markov chain

on S with transition matrix $P^{\frac{1}{m}} := e^{\frac{Q}{m}}$

($\because$ Thm 2.13 $\Rightarrow \frac{Q}{m}$ - Q matrix $\Rightarrow P^{\frac{1}{m}}$ is stochastic matrix)

Define $m \geq 1, n \geq 0$ $X_{\frac{n}{m}} := X_n^{(m)}$ (Assumption $X_0^{(m)} = Y \ \forall m \geq 1$)

$\Rightarrow (X_n)_{n \geq 1} = (X_n^{(n)})_{n \geq 1}$

The distribution of $X^{(m)}_n$: $X^{(m)}_n$ — is determined by Y and $(P^{1/m})^n = (e^{Q/m})^n = e^{nQ/m}$

.... $X^{(n)}_n$ — .... Y .. $e^{nQ/n} = e^{Q} = P$

$\{X_{n/m} : n \geq 0, m \geq 1\} \xrightarrow[\text{Extends}]{?} \{X_t : t \geq 0\}$ $\square$

M.C $[P^{\frac{1}{m}}]^n$

Example 2.1.4 $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$ $P(t) = e^{tQ}$, $p_{11}(t) = ?$

$0 = \det(Q - \lambda I) \Rightarrow \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -1-\lambda & 0 \\ 2 & 1 & -3-\lambda \end{vmatrix} = 0 \Rightarrow$

$$0 = 1 - 2(-1-\lambda) + (-3-\lambda)\left((-2-\lambda)(-1-\lambda) - 1\right)$$

$$0 = 1 + 2 + 2\lambda + (3+\lambda)\left(-(2+\lambda)(1+\lambda) + 1\right)$$

$$0 = 1 + 2 + 2\lambda + (3+\lambda)(-1 - \lambda^2 - 3\lambda)$$

$$0 = \cancel{1+2} + 2\lambda \cancel{-3} - 3\lambda^2 - 9\lambda - \lambda - \lambda^3 - 3\lambda^2$$

$$= -8\lambda - \lambda^3 - 6\lambda^2 = -\lambda(\lambda+2)(\lambda+4)$$

$$\Rightarrow Q = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} U^{-1} \quad \text{for some } U_{3\times 3}.$$

$$\Rightarrow e^{tQ} = e^{tU\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}U^{-1}} = \sum_{k=0}^{\infty} \frac{\left(U\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}U^{-1}\right)^k t^k}{k!} = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-4t} \end{bmatrix} U^{-1}$$

So $p_{11}(t) = a + b e^{-2t} + c e^{-4t}$ for some a, b, c

$$1 = a + b + c \quad (\text{at } t=0)$$

$$-2 = q_{11} = p'_{11}(0) = -2b - 4c$$

$$7 \underset{\substack{\uparrow \\ \text{check}}}{=} q^{(2)}_{11} = p''_{11}(0) = 4b + 16c$$

$$\Rightarrow p_{11}(t) = \frac{3}{8} + \frac{1}{4} e^{-2t} + \frac{3}{8} e^{-4t}$$

Example 2.1.5

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & & \\ 0 & -\lambda & \lambda & 0 & \\ & & \ddots & \ddots & \\ 0 & 0 & \cdots & -\lambda & \lambda \\ & & & & 0 \end{pmatrix}_{N \times N}, \quad P(t) = e^{tQ}$$

$p_{ij}(t$

① p'_{ii}

② p'_{ij}

③ p'_{iN}

① p_{ii}

$p_{ij}(t) = 0 \quad j < i \quad \forall\, t \geq 0$ Q - upper Δar $\Rightarrow$ P is upper Δar

① $p'_{ii}(t) = -\lambda\, p_{ii}(t)$ (Backward equation) , $p_{ii}(0) = 1 \quad i < N$

② $p'_{ij}(t) = -\lambda\, p_{ij}(t) + \lambda\, p_{i,j-1}(t)$, $p_{ij}(0) = 0 \quad i < j < N$

③ $p'_{iN}(t) = \lambda\, p_{i,N-1}(t)$, $p_{iN}(0) = 0 \quad i < N$

① $p_{ii}(t) = e^{-\lambda t} \quad i < N$

$i < j <$

$p_{i,i+1}$ ←

⋮

$p_{i,i+k+1}$ i

Dar

$i < j < N$ $\qquad \frac{d}{dt}x(t) = -\lambda x(t) + y(t) \Rightarrow \frac{d}{dt}\left(e^{+\lambda t}x(t)\right) = y(t)e^{+\lambda t} \Rightarrow x(t) = x_0 + e^{-\lambda t}\int_0^t y(s)e^{-\lambda s}\,ds$

$$\longrightarrow \left(e^{\lambda t}p_{ij}(t)\right)' = \lambda e^{\lambda t}p_{i,j-1}(t)$$

$$p_{i,i+1} \xleftarrow{\quad\quad} \underset{Ex}{\Longrightarrow} \quad p_{ij}(t) = \frac{e^{-\lambda t}(\lambda t)^{j-i}}{j-i\,!} \qquad j < N$$

$\vdots$

$$p_{i,i+k+1} \qquad i = 0, \qquad p_{0j}(t) = \frac{e^{-\lambda t}(\lambda t)^{j}}{j!} \qquad 0 < j < N.$$

$$(Ex) \quad p_{iN}(t) = \lambda\int_0^t e^{-\lambda s}\frac{(\lambda s)^{N-i-1}}{(N-i-1)!}\,ds.$$