

Recall:

$\{X_n\}_{n \geq 0}$ M.C on a state space S , with transition matrix P & initial distribution μ .

$$P(X_n = i_n, \dots, X_0 = i_0) = \mu(i_0) p_{i_0 i_1} \dots p_{i_{n-1} i_n}$$

$$\forall i_j \in S \quad 0 \leq j \leq n \quad \forall n \geq 1$$

$$(S, P, \delta_i) \in \mathcal{P}_i(\cdot)$$

Markov Property

$$P(X_{n-1} = i_{n-1}, \dots, X_0 = i_0) > 0 \text{ for } \sum_{\substack{i_k \in S \\ 0 \leq k \leq n-1}} 1$$

then

$$P(X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_n = i_n | X_{n-1} = i_{n-1})$$

cha

Example 1.3

$V \equiv$ state

$E \subset \{ \dots \}$

$G = (V, E)$

$M: E \rightarrow [0, 1]$

Chapman-Kolmogorov

$$P_i(X_n=j) = \sum_{k \in S} P_i(X_s=k) P_k(X_{n-s}=j) \quad s+s=n$$

$$\left(P_{ij}^{(n)} = \sum_{k \in S} P_{ik}^{(s)} P_{kj}^{(n-s)} \right)$$

Example 1.3.3 [Random walk on graph] $[V, E, \mu]$

$V \equiv$ state space. (vertex set of graph)

$E \subset \{ \{i,j\} \mid i \in V, j \in V, i \neq j \}$ we will say $i \sim j$ if $\{i,j\} \in E$

$G = (V, E) \equiv$ graph. Assume $\# \{j \mid j \sim i\} < \infty \quad \forall i \in S$

$\mu: E \rightarrow [0, \infty)$ such that $\mu(\{i,j\}) = \mu_{ij} \quad \& \quad \mu_{ij} = \mu_{ji} ; \mu_{ij} \neq 0 \Leftrightarrow i \sim j$

$\{X_n\}_{n \geq 0}$

$\mathcal{X}_n, n \geq 0$

$\mathcal{E}$

(a) $V = \mathcal{S}$

μ_i

$$i, j \in V$$

$$p_{ij} = \frac{\mu_{ij}}{\mu_i} \quad \text{where} \quad \mu_i = \sum_{j \in V} \mu_{ij}$$

$\{X_n\}_{n \geq 0}$ is random walk on graph (V, E, μ) ; starting at $i \in S$ if

$\{X_n\}_{n \geq 0}$ is a Markov chain on V , transition matrix $P = [p_{ij}]_{(i,j) \in V \times V}$
 & initial distribution δ_i

(a) $V = \{0, 1, \dots, N\}$

$$\mu_{ij} = \begin{cases} 1 & \text{if } i = j \pm 1, i, j \in V \\ 0 & \text{otherwise} \end{cases}$$

$$E = \{ (i, j) \mid i = j \pm 1 \text{ \& } i, j \in V \}$$

$$p_{ij} = \begin{cases} \frac{1}{2} & i = j \pm 1 \text{ \& } i \neq 0, j \neq N \\ 1 & i = j = 0 \\ 1 & i = j = N \end{cases}$$

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

$\{X_n\}_{n \geq 0}$ simple random
walk on $\{0, 1, \dots, N\}$
reflected at 0 & N .

(b) $V = \mathbb{Z}$ $E = \{ \{i, j\} \mid |i - j| = 1, i, j \in \mathbb{Z} \}$

$$\mu_{ij} = \mu_{ji} = 1 \quad \forall \{i, j\} \in E$$

$$P = [p_{ij}]$$

$$p_{ij} = \begin{cases} \frac{1}{2} & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{if } |i - j| = 1$$

otherwise

$$\{X_n\}_{n \geq 0} \text{ M.C.}$$

on $\mathbb{Z}$, P, μ

is a simple
random walk on $\mathbb{Z}$

Then

Example



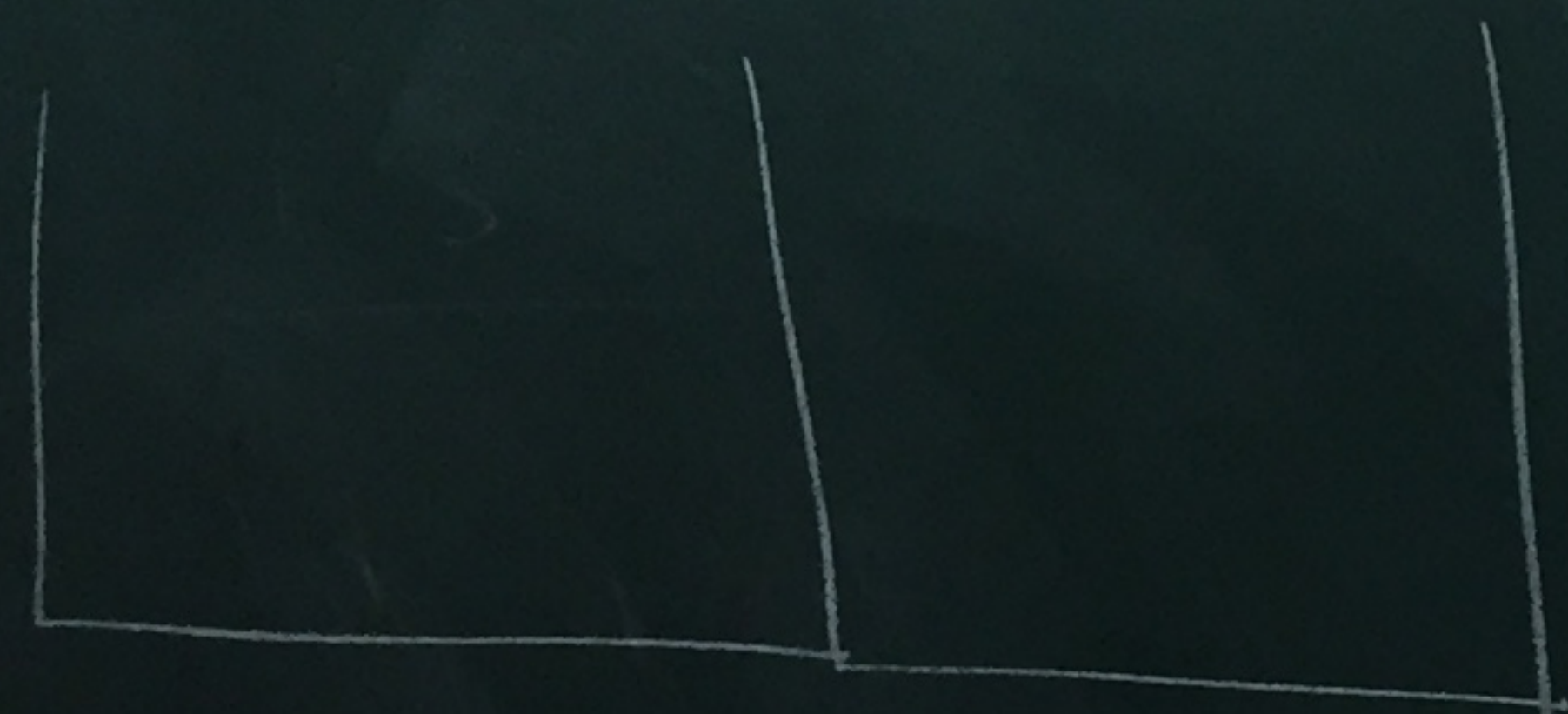
U, v

$2N$

m

These are topics of study by themselves $P(X_n = k) P_{X_n}(X_n = 0)$

Example 1.3.4 [Ehrenfest Model]



- Urn I II - Urn
- $2N$ balls are distributed in the two urns via

μ

Dynamics : At each time step, $n \geq 1$

- choose one of $2N$ balls uniformly
- transfer the ball from its urn to the other urn.

$X_n = \#$ of balls in urn I.

$(X_n)_{n \geq 0}$

with transition matrix

$$P = [P_{ij}]$$

& initial dist

$(X_n)_{n \geq 0}$ is a Markov chain

$$S = \{0, \dots, 2N\}$$

with
transition
matrix

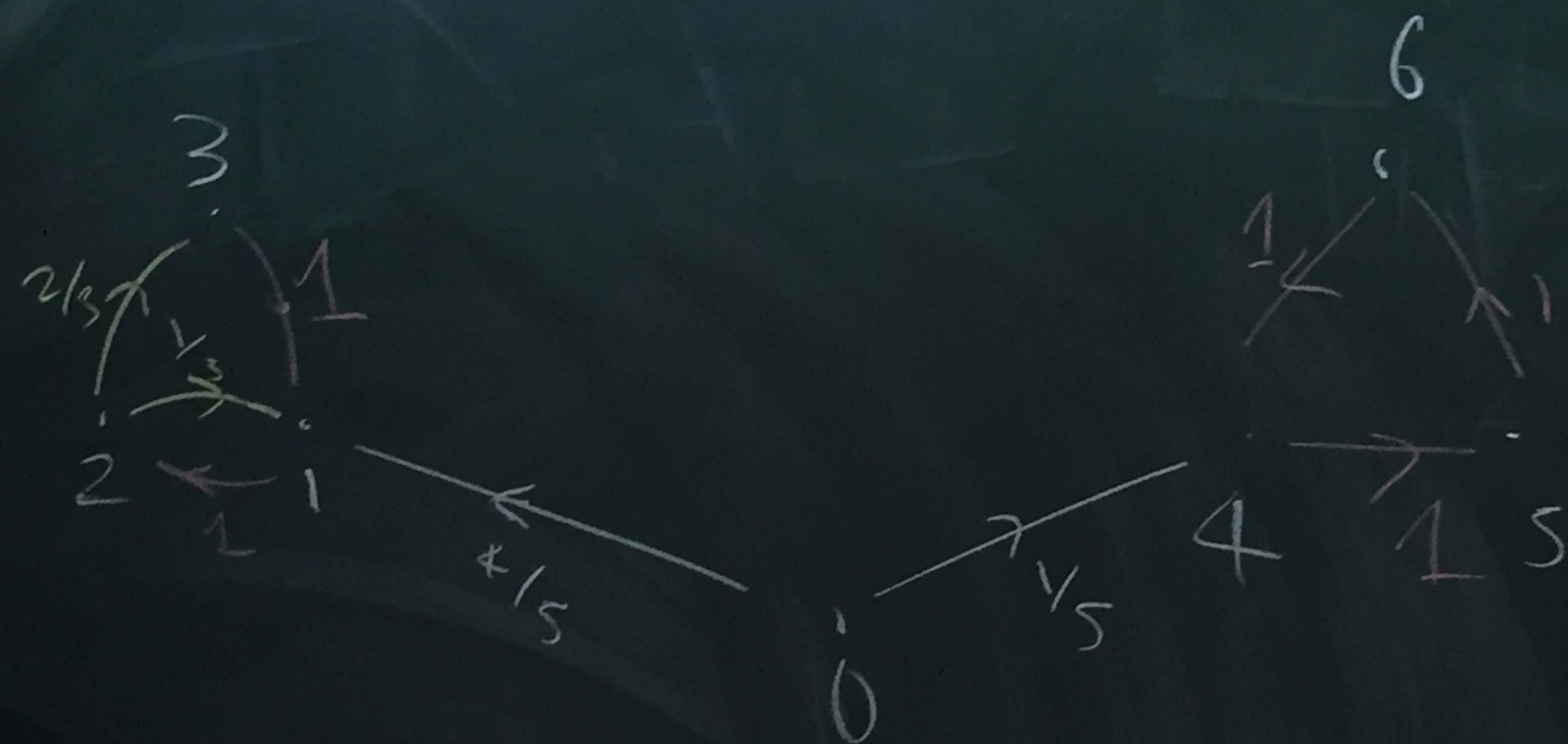
$$P = (P_{ij})$$

$$P_{ij} = \begin{cases} 1/2N & i = j+1, i \neq 0 \\ \frac{2N-i}{2N} & i = j-1, i \neq 2N \\ 1 & i = 0, j = 1 \\ 1 & i = 2N, j = 2N-1 \\ 0 & \text{otherwise} \end{cases}$$

& initial distribution μ .

1.4 Classification of states

Review Example 1.1.2



1.1.1

Formal
differences

$\{1, 2, 3\}$

$\{4, 5, 6\}$

$\{0\}$

Definition

P.

(All)

We say

Definition 1.4.1 : - $(X_n)_{n \geq 0}$ M.C. on state space S , transition matrix

P . We say $i \in S$ leads to $j \in S$

if

$$\mathbb{P}_i (X_n = j \text{ for some } n \geq 0) > 0$$

- denoted by $i \rightarrow j$

(Aliter: - $j \in S$ is accessible from $i \in S$ if $i \rightarrow j$)

We say $i \in S$ communicates with $j \in S$ if i leads to j & j leads to i

denoted by $i \leftrightarrow j$

Remarks : $i \leftrightarrow i$ $P_i(X_0 = i) = 1 \Rightarrow P_i(X_n = i \text{ for } \text{some } n \geq 0) = 1 > 0$

Theorem 1.4.2 : - let $i \neq j$ $i, j \in S$

The following are equivalent

(i) $i \rightarrow j$

(ii) $\exists n \geq 1$ $i_k \in S$ $k \in \{1, \dots, n-1\}$
 such that $p_{i_1 i_2} \dots p_{i_{n-1} j} > 0$

(iii) $\exists n \geq 1$ such $p_{ij}^{(n)} > 0$

Proof:

Observe, $n \geq 1$

$$p_{ij}^n = P_i(X_n = j) \stackrel{\text{(subset)}}{\leq} P_i(X_k = j \text{ for some } k \geq 0)$$

$$= P_i\left(\bigcup_{k=1}^{\infty} \{X_k = j\}\right)$$

$$\stackrel{\text{(Union bound)}}{\leq} \sum_{k=1}^{\infty} P_i(X_k = j)$$

$$\stackrel{\text{(def)}}{=} \sum_{k=1}^{\infty} p_{ij}^{(k)}$$

(i) (=) (iii)

By definition

$$n \geq 1 \quad p_{ij}^n = \sum_{\substack{i_k \in S \\ 1 \leq k \leq n-1}} p_{ii} p_{i_1 i_2} \dots p_{i_{n-1} j} \quad \boxed{(ii) \Rightarrow iii)} \quad \square$$

Equivalence Relation $(S, \leftrightarrow)$

• [Reflexivity] $i \leftrightarrow i$ or $p_{ii}^0 > 0 \quad \forall i \in S$

• [Transitivity] $i \leftrightarrow j$ & $j \leftrightarrow k \xRightarrow{\text{Thm 14.2 (iii)}} \exists r \geq 1, s \geq 1, p_{ij}^r > 0 \text{ \& } p_{jk}^s > 0$
 $\exists m \geq 1, t \geq 1, p_{ji}^m > 0 \text{ \& } p_{kt}^t > 0$

• [Symmetry]

Chapman Kolmogorov equation $\Rightarrow$

$$\begin{aligned} & p_{ik}^{r+s} \geq p_{ij}^r p_{jk}^s > 0 \Rightarrow i \rightarrow k \\ \& \quad p_{ki}^{l+m} \geq p_{kj}^l p_{ji}^m > 0 \Rightarrow k \rightarrow i \end{aligned}$$

$$i \leftrightarrow k$$

[Symmetric] $i \leftrightarrow j \Leftrightarrow i \rightarrow j \Rightarrow j \rightarrow i \Leftrightarrow j \leftrightarrow i$

Example 112

do $\{4,5,6\}, \{1,2,3\}$
are communicating classes

Definition 1.4.35. - S can be partitioned into
equivalence classes called communicating classes
by the equivalence relation $P \Leftrightarrow$.

(ii) We say a communicating class, C , is closed
if $i \in C \ \& \ i \rightarrow j \implies j \in C$

(iii) A state $i \in S$ is said to be absorbing state
if $\{i\}$ is a closed communicating class

(iv)

in a

Exam

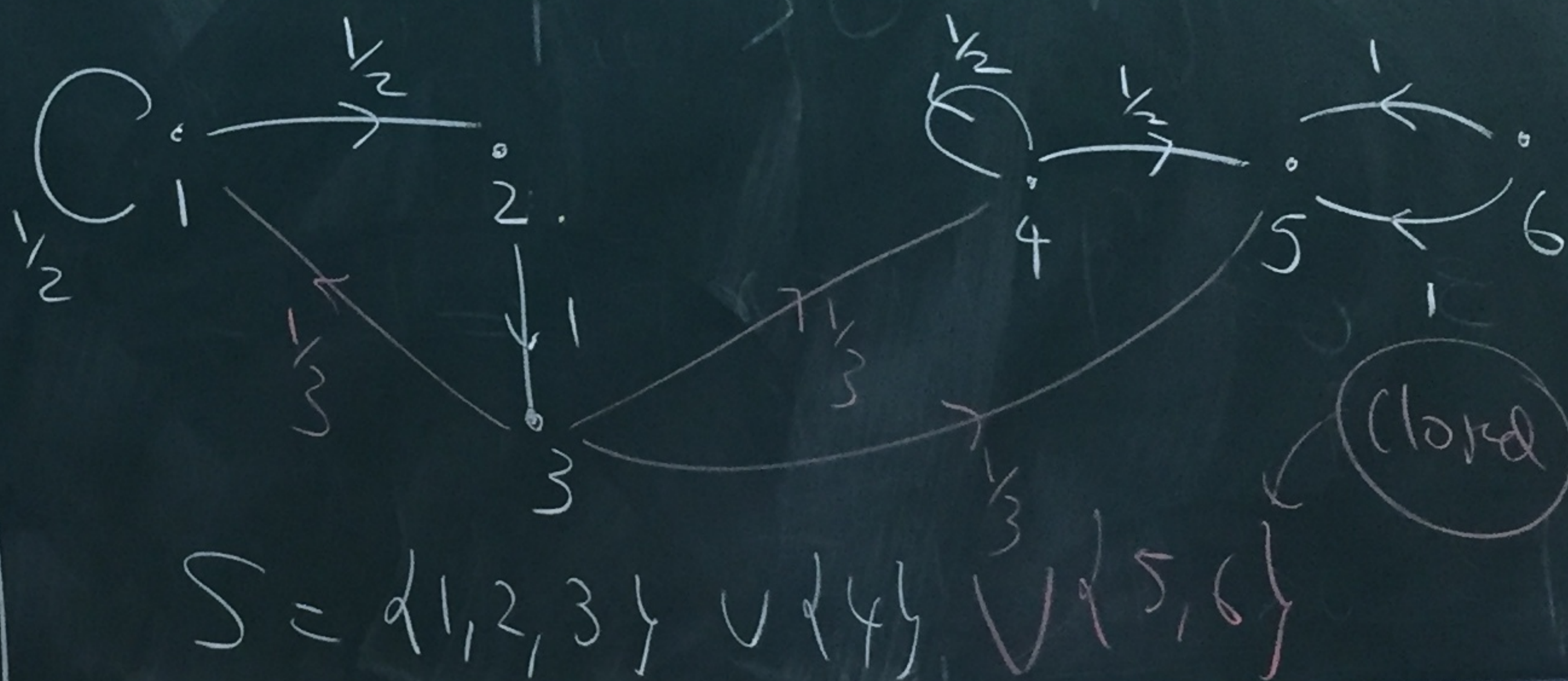
$P =$ [

(IV) If S is a single closed communicating class then the Markov chain is said to be irreducible.
 (ie $i, j \in S \Rightarrow \exists n \geq 1, P_{ij}^{(n)} > 0$)

Example 1.4.4:-

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



1.5 Hitting time & Absorption Probabilities

$$(X_n)_{n \geq 0}$$

be a Markov chain
on $(\Omega, \mathcal{F}, \mathbb{P})$

with $\left\{ \begin{array}{l} \text{state space } S \\ \text{transition matrix } P \\ \text{initial distribution } \mu \end{array} \right.$

$$\text{let } A \subseteq S \quad T_A: \Omega \rightarrow \{0\} \cup \mathbb{N} \cup \{\infty\}$$

$\#$
 $\emptyset$

defined by

$$T_A(\omega) = T_A = \inf \{ n \geq 0 \mid X_n \in A \}$$

$\equiv$
 $X_n(\omega)$

Convention :- $\inf \{\emptyset\} = \infty$

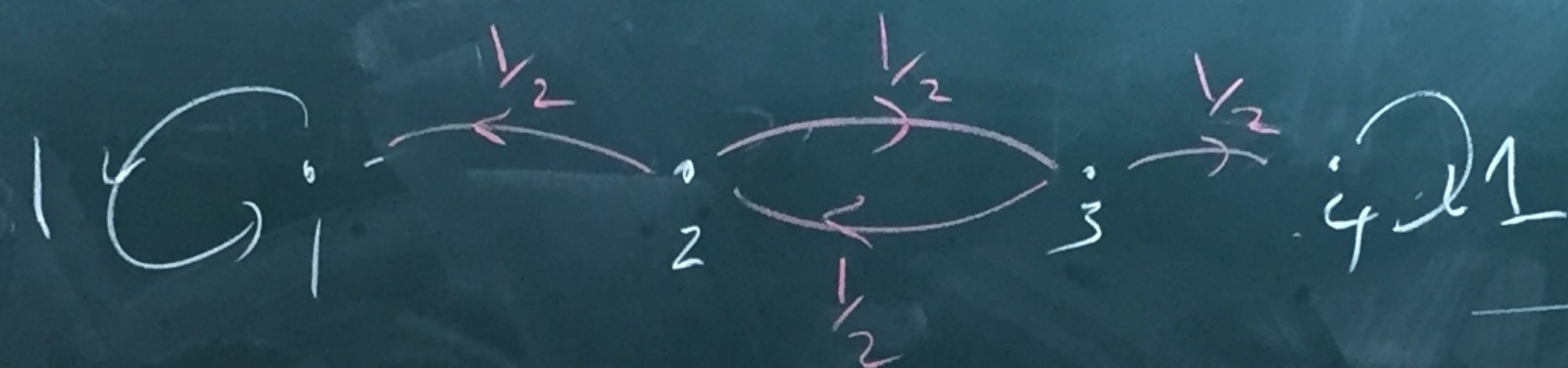
Def 1.4.35 Let S be a Markov chain with K states. Partition S into communicating classes A and B . Let $h^A: S \rightarrow [0,1]$ be such that

$$h^A(i) = \mathbb{P}_i(T_A < \infty)$$

i.e. $h^A(i)$ is the probability that the Markov chain starting at i ever hits A .

• If A is a closed communicating class then then $h^A(i) \equiv 1$ absorption probability of A starting at i .

(IV) Example 1.5.1 $(X_n)_{n \geq 0}$ M.C. on $\{1, 2, 3, 4\}$ given by



$$A = \{4\}$$

Find $h^A: S \rightarrow [0, 1]$

$$T_A = \inf \{n \geq 0 \mid X_n = 4\}$$

$$h^A(4) = \mathbb{P}_4(T_A < \infty) = \mathbb{P}_4(T_A = 0) = 1$$

$$h^A(1) = \mathbb{P}_1(T_A < \infty) = 0$$

$$(\mathbb{P}_1(X_n = 4 \text{ for some } n \geq 0) = 0)$$

Intuitively

$$h^A(2) = \frac{1}{3}$$

2

$$h^A(3) = 2/3$$

$$h^A(2) = \frac{1}{2} h^A(3) + \frac{1}{2} h^A(1)$$

$$\Rightarrow h^A(2) = \frac{1}{2} h^A(3)$$

$$h^A(3) = \frac{1}{2} h^A(2) + \frac{1}{2} h^A(4)$$

$$\Rightarrow h^A(3) = \frac{1}{2} h^A(2) + \frac{1}{2}$$