

2 Continuous time Markov chains $\{X_t\}_{t \geq 0}$ $X_t \in S, S$ countable

Definition 2.1.1: Let S be a countable set

A Q -matrix on S is matrix $Q = [q_{ij}]_{i,j \in S}$

satisfying the following:-

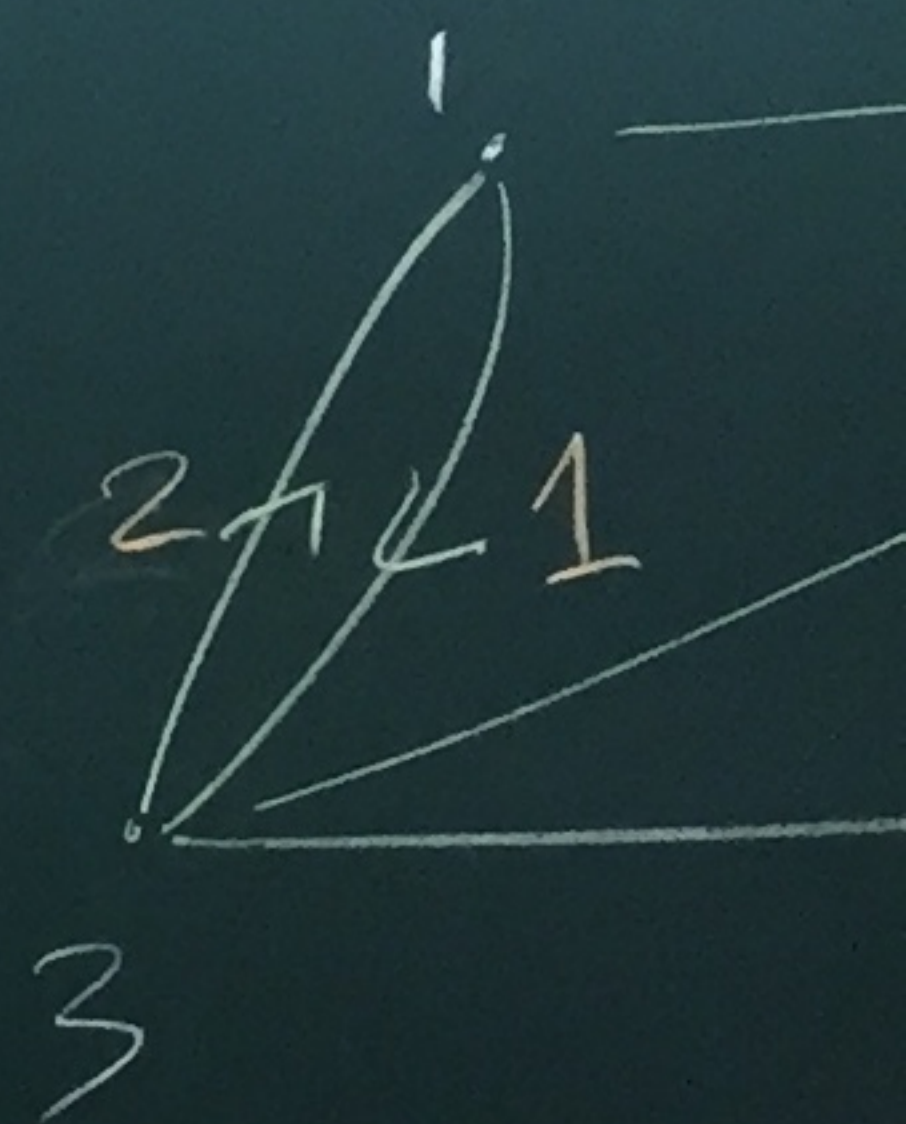
(i) $0 \leq -q_{ii} < \infty$ for all $i \in S$

(ii) $q_{ij} \geq 0$ for all $i \neq j, i, j \in S$

(iii) $\sum_{j \in S} q_{ij} = 0$ for all $i \in S$

$$q_{ii} = -\sum_{j \neq i} q_{ij} < \infty$$

Graphical



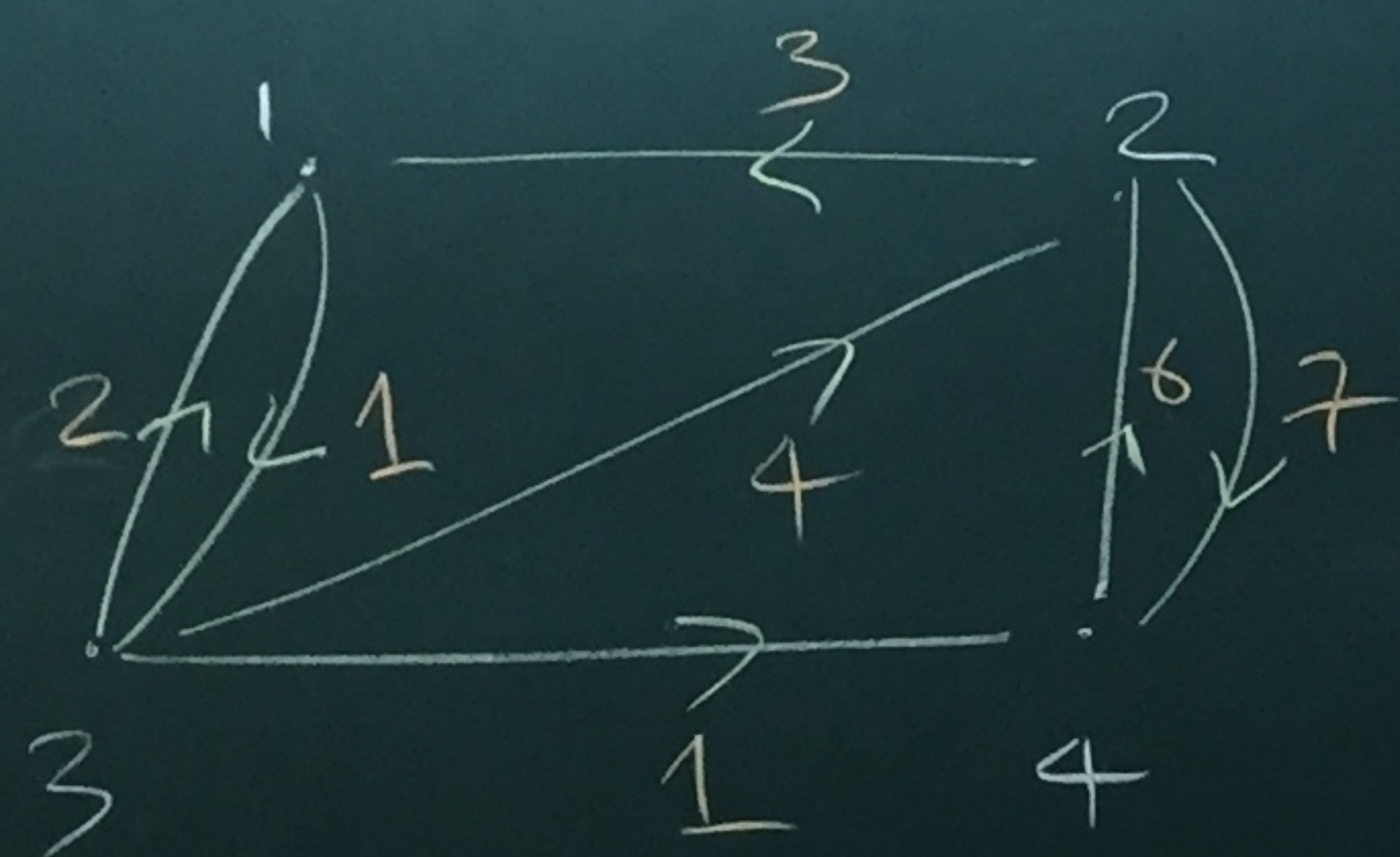
Interpretation

To make a b...

table

edges

Graphical representation for Q on S



$$Q = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 3 & -10 & 0 & 7 \\ 2 & 4 & -7 & 1 \\ 0 & 6 & 0 & -6 \end{bmatrix}$$

Interpretation : $Q_{ij} \equiv$ "rate at which" go from i to j

- Think of time spent at site $i \sim \frac{1}{\text{rate}} \sim \frac{1}{-Q_{ii}}$
- "jump to site j " with probability $\frac{Q_{ij}}{-Q_{ii}}$

To make above interpretation rigorous, we need to connect Q to a discrete Markov chain on S _{time}

$$\sum_{j \neq i} Q_{ij} < \infty$$

Suppose

$\mathbb{N} \cup \{0\} \hookrightarrow \mathbb{R}_+$ embedded in the classical inclusion manner.

$p \in (0, \infty)$

$$q = \log p$$

$$\{p^n, n \geq 0\} \xleftrightarrow{1-1} \{e^{nq}, n \geq 0\} \xrightarrow{\text{extend to } \mathbb{R}_+} \{e^{tq}, t \geq 0\}$$

let us assume $|S| < \infty$ and carry the above intuition to $\mathbb{Q}$

$\forall k \geq 0$ Q^k is well-defined, $Q^0 = I$

$$\Rightarrow A_n = \sum_{k=0}^n \frac{Q^k}{n!} = \left[\sum_{k=0}^n \frac{q_{ij}^k}{n!} \right]_{i,j \in S}$$

$$\text{Suppose } \sum_{k=0}^n \frac{q_{ij}^k}{n!} \rightarrow a_{ij} \Rightarrow A_n \rightarrow A \text{ \& we set } A = e^Q.$$

in on $\mathbb{P}$.

Question: - Given a transition matrix $P = [P_{ij}]$ is there
a Q matrix such that $P = e^{Q}$? ^{countable}

It answer is yes then -

$$\left\{ P^n : n \geq 0 \right\} \xleftrightarrow[\text{onto}]{\text{1-1}} \left\{ e^{nQ} : n \geq 0 \right\} \xrightarrow[\text{to } \mathbb{R}_+]{\text{extend}} \left\{ e^{tQ} : t \geq 0 \right\}$$

$\parallel$?
 discrete time Markov chain on S continuous time version $\{X_t\}_{t \geq 0}$ of $\{X_n\}_{n \geq 0}$

Observe

$$Q_1 \& Q_2$$

$$e^{Q_1 + Q_2} = e^{Q_1} e^{Q_2}$$

$\Uparrow$
 $Q_1 Q_2 = Q_2 Q_1$

$$P =$$

For $t =$

Theorem 2.1.2

Set

$$\{P(t) : t \geq 0\}$$

$$(i) \quad P(t)$$

$$(ii) \quad \{P(t)\}$$

(Equation should component)

$$P = e^Q, \quad P^n = (e^Q)^n = e^{nQ}$$

For $t \geq 0$, set $P(t) = e^{tQ}$, $P(0) = I$

Theorem 2.1.2 : Let Q be a matrix on S & $|S| < \infty$.

Set $P(t) = e^{tQ}$ $t \geq 0$ & $P(0) = I$. Then

$\{P(t) : t \geq 0\}$ has the following properties :-

$$(i) \quad P(t+s) = P(t)P(s) \quad \forall t, s \geq 0$$

(ii) $\{P(t) : t \geq 0\}$ is UNIQUE solution to the forward equation

(Equation should be picked component wise) $\frac{d}{dt} P(t) = P(t)Q$ & $P(0) = I$

(iii) $\{P(t), t \geq 0\}$ is the unique solution to the backward equation

$$\frac{d}{dt} P(t) = Q P(t), \quad P(0) = I.$$

(iv) for all $k \in \mathbb{Z} \cup \mathbb{N}$.

$$\left. \frac{d^k}{dt^k} P(t) \right|_{t=0} = Q^k$$

Proof:-

$|S| < \infty$ & $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $= \infty$

$$a_j = \max_{i,j \in \mathbb{Z}} |a_{ij}|$$

$$\left| \sum_{n=0}^{\infty} \frac{a_{ij}^n}{n!} \right| \leq \sum_{n=0}^{\infty} \frac{a_{ij}^n}{n!} = \infty$$

$\Rightarrow e^{tQ}$ is well defined for all $t \geq 0$.

Question: Given $s, t \in \mathbb{R}_+$ sQ & tQ commute.

$$\Rightarrow e^{sQ} e^{tQ} = e^{(s+t)Q}$$

$$\Rightarrow P(s)P(t) = P(s+t)$$

So we have shown (i).

$$\text{Now, } P(t) = e^{tQ}$$

$$p_{ij}(t) = \sum_{k=0}^{\infty} t^k \frac{q_{ij}^k}{k!}$$

$$[q_{ij}^k] = Q^k$$

[Real Analysis facts] $\Rightarrow$

$$\frac{d}{dt} P(t)$$

$$\Rightarrow \frac{d}{dt}$$

$\Rightarrow$

let $M(t)$

$$\frac{d}{dt} p_{ij}(t) = \sum_{k=1}^{\infty} t^{k-1} \frac{q_{ij}^k}{(k-1)!} \quad \forall (i,j) \in S$$

$$\Rightarrow \frac{d}{dt} P(t) = \sum_{k=1}^{\infty} t^{k-1} \Phi^k = \Phi P(t) = P(t) \Phi$$

$$\Rightarrow \frac{d}{dt} P(t) = P(t) \Phi \quad \& \quad \frac{d}{dt} P(t) = \Phi P(t)$$

$$P(0) = I \quad P(0) = I$$

Let $M(t)$ be another solution to

$$\frac{d}{dt} M(t) = \Phi M(t) \quad \& \quad M(0) = I \quad - (*)$$

(i) Set

Exercise

(iv)

=)

Similarly

$$(iii) \text{ Set } H(t) = e^{-tQ} M(t)$$

Exercise: $\frac{d}{dt} H(t) = e^{-tQ} (-Q) M(t) + e^{-tQ} \frac{dM(t)}{dt}$

$$(iv) \quad e^{-tQ} H(t) = e^{-tQ} \left[-Q M(t) + \frac{dM(t)}{dt} \right] = 0 \quad \text{as } H(t) \text{ solves } (*)$$

$$\Rightarrow H(t) = H(0) = I$$

$$\Rightarrow e^{-tQ} M(t) = I \Rightarrow M(t) = e^{tQ} \quad \text{so } M = P$$

Similarly one can show $\frac{dP}{dt} = P(t)Q$, $P(0) = I$ has a unique solution

Question: Prove that (ii) & (iii) are equivalent.

For (ii), observe

$$\frac{d^m}{dt^m} p_{ij}(t) = \sum_{k=m}^{\infty} \frac{t^{k-m}}{(k-m)!} q_{ij}^k$$

At $t=0$

$$\left. \frac{d^m}{dt^m} p_{ij}(t) \right|_{t=0} = q_{ij}^m$$

$$\Rightarrow \left. \frac{d^m}{dt^m} p_{ij}(t) \right|_{t=0} = q_{ij}^m \quad \forall m \geq 1$$

□

Theorem

Proof:

$i \neq j$
 $i, j \in S$

$|S| < \infty$

Theorem 2.1.3 - A matrix Q on a finite set S is

Q -matrix $\iff P(t) = e^{tQ}$ is a stochastic matrix $\forall t \geq 0$.

Proof:- let Q be Q -matrix

$$i \neq j, \quad i, j \in S \quad p_{ij}(t) = 0 + a_{ij}t + \sum_{k=2}^{\infty} \frac{a_{ij}^k}{k!} t^k$$

$$|S| < \infty \quad a = \max_{i,j \in S} |a_{ij}| \Rightarrow \left| \sum_{k=2}^{\infty} \frac{a_{ij}^k}{k!} t^k \right| \leq \sum_{k=2}^{\infty} \frac{(|S|a^2)^k}{k!} t^k = \left(\sum_{k=0}^{\infty} \frac{(|S|a^2)^k}{(k+2)!} t^k \right) (|S|a^2)^2 t^2 \leq C_1 t^2 \quad 0 < t < 1$$

(i) $0 < \dots$
 $\Rightarrow \exists$

$$\delta = \frac{a_{ij}}{c_1 + (-c_1 t + a_{ij})} \quad \forall t < \dots$$

Fix $t > 0$,

$$t \geq 0$$

$$(ii) \quad 0 < t < 1, \quad i \neq j$$

$$-c_1 t^2 + a_{ij} t$$

$$\text{If } a_{ij} > 0$$

$$\Rightarrow \exists \delta > 0 \text{ such that}$$

$$\delta = \frac{a_{ij}}{c_1 + 1}$$

$$(-c_1 t + a_{ij})t \geq 0$$

$$\forall t < \delta$$

$$-c_1 t^2 + a_{ij} t \leq p_{ij}(t) \leq a_{ij} t + c_1 t^2$$

$$0 \leq p_{ij}(t) \quad \forall \quad 0 < t < \delta$$

$$\text{Now observe, } P(t) = e^{tQ}$$

$$\forall n \geq 1$$

$$t > 0$$

$$= \left(e^{t/n Q} \right)^n = P(t/n)^n \quad \text{--- } (+)$$

$$\text{Fix } t > 0, \text{ choose } n \text{ large } \frac{t}{n} < \delta \Rightarrow p_{ij}(t/n) \geq 0 \xrightarrow{(+)} p_{ij}(t) \geq 0$$

If $p_i(t) \geq 0$

$$(\dagger) \Rightarrow a_i t \geq p_i(t) - c_i t^2$$

Take $t > 0$ & small enough so that

$$p_i(t) - c_i t^2 \geq 0 \quad - \textcircled{?}$$

$$\Rightarrow a_i t \geq 0$$

$$\Rightarrow a_i \geq 0$$

gap $- a_i \geq 0 \Leftrightarrow p_i(t) \geq 0 \quad \forall t > 0$

$$(i) \quad |s| < \dots$$

$$\sum_{i \in S} p_i(t)$$

$$\sum_{k \in S} a_{ik} = 0 \Rightarrow$$

$$(|S| < \infty)$$

$$\sum_{k \in S} a_{ik}^n = \sum_{k \in S} \sum_{l \in S} a_{il}^{n-1} a_{lk}$$

$$= \sum_{l \in S} a_{il}^{n-1} \left(\sum_{k \in S} a_{lk} \right) = 0$$

$$\Rightarrow p_i(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} a_{ii}^n$$

$$\sum_{i \in S} p_i(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i \in S} a_{ii}^n = 1 + 0 = 1$$

$$\sum_{i \in S} p_i(t) = 1 \quad \forall t \geq 0 \text{ then,}$$

$$a_{ii} = \left. \frac{d}{dt} p_i(t) \right|_{t=0} \quad (\text{Thm 2.12 (iv)})$$

$$(11) \quad |S| < \infty \Rightarrow$$

$$\sum_{j \in S} a_{ij} = \sum_{j \in S} \left. \frac{d}{dt} p_{ij}(t) \right|_{t=0}$$

$$= \left. \frac{d}{dt} \left(\sum_{j \in S} p_{ij}(t) \right) \right|_{t=0}$$

$$= \left. \frac{d}{dt} (1) \right|_{t=0}$$

$$= 0$$

$$a_{ij} > 0 \Rightarrow p_{ij}(t) > 0 \quad (i \neq j)$$

$$\sum_{j \in S} a_{ij} = 0 \Leftrightarrow \sum_{j \in S} p_{ij}(t) = 1 \quad \forall t \geq 0$$