Due : Tuesday, October 21st 2003

1. Supandi is often late for work, although not consistently. His behaviour depends on what happened the previous two days. If he was on time today and yesterday, then he will be late tomorrow with probability 0.4. If he was on time today but late yesterday, then he will be late tomorrow with probability 0.5. If he was on late today but on time yesterday, then he will be late tomorrow with probability 0.6. Finally if he was late today and yesterday then he puts in a special effort and he will be late tomorrow with probability 0.1 only. Let X_n denote L if he was late and O if he is on time.
   (a) Is $\{X_n : n \geq 0\}$ a Markov Chain ? If yes, then write down its transition matrix, P.
   (b) Let $Y_n = (X_{n-1}, X_n)$. Is $(Y_n, n \geq 0)$ a Markov Chain ? If yes, write down its transition matrix P.

2. Consider the Markov Chain describing the position of a particle on graph with eight vertices, having transition matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

   (a) Assuming that the particle picks each edge(need not be a straight line) equally likely. Sketch the graph with 8 labeled vertices.
   (b) Determine the closed[1] communicating classes and their periodicity.
   (c) Classify the states of the chain into transient and recurrent.

3. Consider a Markov chain with state space $\{A, B, C\}$ and the following transition matrix

$$P = \begin{pmatrix} .2 & .4 & .4 \\ .4 & .4 & .2 \\ .4 & .6 & 0 \end{pmatrix}.$$

   (a) what is the probability of going from state A to state B in one step ?
   (b) what is the probability of going from state B to state C in exactly two steps ?
   (c) what is the probability of going from state C to state A in exactly two steps ?
   (d) what is the probability of going from state C to state A in exactly three steps ?
   (e) Calculate the second, third and fourth power of this matrix. Do you have a guess for P^n for n large.

Problems to be turned in are: 2,4

[1] A communicating class C is said to be closed if it is impossible to get out i.e. $P_{ij} = 0$ for $i \in C$ and $i \notin C$

4. Consider a Markov chain on $\{1, 2, 3 \ldots\}$ described by the following transition matrix

$$\begin{pmatrix} p_1 & (1-p_1) & 0 & 0 & \cdots & 0 & \cdots \\ p_2 & 0 & (1-p_2) & 0 & \cdots & 0 & \cdots \\ p_3 & 0 & 0 & (1-p_3) & \cdots & 0 & \cdots \\ \vdots & & & \cdots & & \cdots & \cdots \\ p_m & \cdots & & \cdots & (1-p_m) & \cdots & \cdots \\ \vdots & & & \cdots & & \cdots & \cdots \end{pmatrix},$$

where $0 < p_m < 1$ for all m.

(a) Show that the chain is irreducible and aperiodic.

(b) Identify the necessary and sufficient condition for the chain to be recurrent via the following steps:

i. Show $f_{11}^n = p_n \prod_{m=1}^{n-1}(1-p_m)$

ii. Show $f_{11}^n = u_{n-1} - u_n$, where $u_n = \prod_{m=1}^{n}(1-p_m)$.

iii. Decide when $u_n \to 0$ as $n \to \infty$.

5. In a car-leasing company's study of its customers who lease a new car every year, it is found that 30% of those who leased maruti 800 the preceding year changed to zen, where as 40% of those who had leased zen changed to Maruti 800. The rest of the customers just repeated the type of car the year before. Let X_n be the type of car leased out by a typical customer.

(a) Write down the transition matrix for the Markov Chain X_n.

(b) Determine the eigen values and eigen vectors of this matrix.

(c) What is the significance of the eigen value 1 and the corresponding eigen vector ? Discuss how this may help the car-leasing company.