Due : Tuesday, October 7th 2003

1. (Ferguson page 124)Find the likelihood equations and the assymptotic distribution of the MLE for the parameters of the gamma distribution $\mathcal{G}(\alpha, \beta)$,

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-\frac{x}{\beta}) I(x > 0),$$

$$\Theta = \{(\alpha, \beta) : \alpha > 0, \beta > 0\}$$

[*You can leave your solutions in terms of* $\frac{d}{d\alpha} \log \Gamma(\alpha)$ *and* $\frac{d}{d\alpha}\Gamma(\alpha)$]

2. (Ferguson page 124) Find the likelihood equations and the assymptotic distribution of the MLE for the parameters of

$$f(x \mid \theta_1, \theta_2) = \exp(-\theta_2 cosh(x - \theta_1) - \zeta(\theta_2)),$$

where the parameter space is $\Theta = \{(\theta_1, \theta_2) : \theta_2 > 0, \}$ and where $\zeta(\theta_2) = \log \int_{-\infty}^{\infty} exp(-\theta_2 cosh(x))dx$.

3. (Lehmann page 564) Determine the rejection region of the Wald, Rao, and likelihood ratio tests of the following hypotheses against two-sided alternatives:

   (a) $H_0 : \lambda = \lambda_0$ when the X are i.i.d. with Poisson distribution $P(\lambda)$.

   (b) $H_0 : p = p_0$ when the X are independent with $P(X_i = 1) = p$, $P(X_i = 0) = 1 - p$

   (c) $H_0 : \sigma^2 = \sigma_0^2$ when the X are i.i.d. $N(0, \sigma^2)$.

4. Let $X_1, X_2, \ldots$ be i.i.d $N(0, 1)$. Suppose that only values $\theta \geq 0$ are possible and that we wish to test $H_0 : \theta = 0$ against $\theta > 0$. Then calculate the MLE $\hat{\theta}_n$. Discuss the convergence of $\sqrt{n}(\hat{\theta}_n - \theta)$.

5. Let $X_1, \ldots X_n$ be i.i.d. according to the distribution with density function

$$f_\theta(x) = \theta \frac{c^\theta}{x^{\theta+1}}, 0 < c < x, 0 < \theta.$$

   (a) Determine the unique solution $\hat{\theta}_n$ of the likelihood equation and find the limit distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$.

   (b) Determine the Wald, Rao and likelihood ratio tests of $H_0 : \theta = \theta_0$ against $\theta \neq \theta_0$

6. Suppose $T_n \to \sigma^2$ and X_n is AN$(0, \sigma^2)$, show that $\frac{X_n}{T_n} \xrightarrow{d} N(0, 1)$

7. Assume (C1)-(C7). Let $\hat{\theta}_n$ be a consistent root of the likelihood equation, the null distribution of $2\Delta_n = 2(l_n(\hat{\theta}_n) - l_n(\theta_0))$ tends to a χ^2 with one degree of freedom.

8. Fill in all the details in Example 7.7.1-5 in Lehmann.

9. Read Theorem 7.7.4 and apply it to Example 7.7.6 in Lehmann.

Problems to be turned in are: 1,2,4,5,6