Due : Thursday, September 11th 2003

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Cauchy with p.d.f $\frac{a}{\pi}\frac{1}{x^2+a^2}, -\infty < x < \infty, a > 0$. Show that
   (a) the likelihood equation has a unique solution $\hat{a}_n$.
   (b) the solution $\hat{a}_n$ corresponds to a local maximum of $l_n(a)$
   (c) $\hat{a}_n$ is the MLE.
2. Let $X_1, \ldots, X_n$ be i.i.d. according to the normal distribution $N(\mu, \sigma^2)$. Assume $\mu = a\sigma, 0 < \sigma < \infty$.
   (a) Write down the log-likelihood function for σ and then proceed to derive the likelihood equation.
   (b) Obtain the MLE for σ.
   (c) Compute $I(\sigma)$.
   (d) Describe the assymptotic distribution of $\hat{\sigma}_n$. [You may assume (C6) and (C7) stated in class are satisfied]
   (e) Observe that $\frac{\bar{X}_n}{a}$ and $\sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n-1}}$ are also estimates of σ and compute their assymptotic distributions.
   (f) Compare the relative efficiency of the above estimates.
3. Let $X_1, \ldots, X_n$ be i.i.d. according to the uniform distribution $U(0, \theta)$.
   (a) Calculate the MLE for θ.
   (b) Show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} 0$
   (c) Compute $I(\theta)$
   (d) Conclude that all of $(C1) - C(5)$ cannot hold for this example. Identify atleast one of them.
4. Let $X_1, X_2, \ldots, X_n$ be i.i.d according to a Cauchy distribution with density $f_\theta(x) = \frac{1}{\pi(1+(x-\theta)^2)}$.
   (a) Find a consistent estimator for θ.
   (b) Calculate δ_n (discussed in class).
   (c) Compute the assymptotic relative efficiency between the above two.
5. Suppose we have a sample of size d from each of n normal populations with commmon unknown variance but possibly different unkown means.

$$X_{ij} \stackrel{d}{=} N(\mu_i, \sigma^2), i = 1, \ldots n, \quad j = 1, \ldots d.$$

   where all the X_{ij} are independent.
   (a) Find the MLE of σ^2.
   (b) Show that for d fixed, $\hat{\sigma}_n^2$ is not consistent for σ^2.

Problems to be turned in are: 2,5