Due : Tuesday, August 26th 2003

1. Suppose a sequence of random variables X_n converges in distribution to $N(\mu, \sigma^2)$. Show that $\frac{X_n - \mu}{\sigma}$ converges in distribution to $N(0,1)$

2. (Ex. 2 Page 42, Ferguson) Let X_n and Y_n be a sequence of independent random variables. Let $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$. Show that $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} X \\ Y \end{pmatrix}$.

3. Prove or disprove the following: If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} Y$, then $X_n + Y_n \xrightarrow{d} X + Y$. [*Hint: Problem 3 in hw2*]

4. Let X_n be $AN(\mu, \sigma_n^2)$ and let
$$Y_n = \begin{cases} 0 & \text{w.p. } 1 - \frac{1}{n}, \\ n & \text{w.p. } \frac{1}{n}. \end{cases}$$
Show that $\frac{Y_n}{\sigma_n} \xrightarrow{p} 0$ and then conclude $X_n + Y_n$ is $AN(\mu, \sigma_n^2)$.

5. (Ex. 3, Page 42, Ferguson) Consider the autoregressive scheme, $X_n = \beta X_{n-1} + \epsilon_n$, where ϵ_i are i.i.d, with mean μ and variance σ^2, $-1 < \beta < 1$ and $X_0 = 0$. Show that $\sqrt{n}(\bar{X}_n - \frac{\mu}{1-\beta}) \xrightarrow{d} N(0, \frac{\sigma^2}{(1-\beta)^2})$, where $\bar{X}_n := \frac{X_1 + X_2 + \ldots X_n}{n}$.

6. (Ex. 4, Page 49, Ferguson)Let $X_1, \ldots, X_n$ be a sample of size n from the beta distribution, $B(\theta, 1), \theta > 0$. The method-of-moments estimate for θ is $\hat{\theta}_n = \frac{\bar{X}_n}{1 - \bar{X}_n}$. Using C.L.T find the assymptotic distribution of $\bar{X}_n$. Now find the assymptotic distribution of $\hat{\theta}_n$.

7. Let $X_1, X_2, \ldots$.. be an i.i.d. sequence of random variables s.t. $E(X_1) = \mu$ and $V(X_1) = \sigma^2 > 0$. Assume all moments of X_i are finite. We want to estimate the coefficient of variation $\frac{\sigma}{\mu}$, using the statistic $T_n := \frac{\sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2}}{\bar{X}_n}$.
   (a) Find the assymptotic distribution of $V_n = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n X_i \\ \frac{1}{n}\sum_{i=1}^n X_i^2 \end{pmatrix}$.
   (b) Find a suitable g such that $g(V_n) = T_n$
   (c) Obtain the assymptotic distribution of T_n.

8. Prove the Claim stated in class: $h(T_n) \xrightarrow{p} 0$.
   (a) Show that $\sqrt{n}(T_n - \mu)$ is $O_p(1)$
   (b) Conclude that $T_n \xrightarrow{p} \mu$ to finish the proof.

9. Let $X_1, X_2, \ldots, X_n$ be i.i.d. F, samples of size n. Let the sample distribution function be $F_n(x) = \frac{\sum_{i=1}^n 1(X_i \le x)}{n}$
   (a) Fix an x. Find the mean and the variance of $F_n(x)$.
   (b) Fix an x. What is the distribution of $F_n(x)$?

Problems to be turned in are: 2,4,7,9