Due : Tuesday, August 19th 2003

1. Let $Y_1,\ Y_2,\ \cdots,\ Y_n$ be independent random variables, each uniformly distributed over the interval $(0,\theta)$. Show that $\max\{Y_1,\cdots,Y_n\}$ converges in probability toward θ as $n\to\infty$.

2. Let $a_n=\sum_{k=0}^{n}\frac{n^k}{k!}e^{-n}$, $n\geq 1$. Using the Central Limit Theorem evaluate $\lim_{n\to\infty}a_n$.

3. Let $Y \stackrel{d}{=} N(0,1)$. Let $X_n=(-1)^nY$. Discuss convergence a.e, in probability, and in distribution of X_n.

4. Let Y be a discrete random variable taking values in $\mathbb{N}$. Show that
   (a) $E(Y)=\sum_{n=1}^{\infty}P(Y\geq n)$.
   (b) Assume $E(Y)=\infty$. Let Y_n be i.i.d copies of Y. Let $X_n=\frac{Y_n}{n}$. Show that $X_n\stackrel{p}{\to}0$ but X_n does not converge a.e.

5. If Y_n is $O_p(\frac{1}{\sqrt{n}})$ then show that Y_n is $o_p(1)$.

6. For $n\geq 1$, let $0\leq p_n\leq 1$ and $\lim\limits_{n\to\infty}p_n=0$. Consider

$$X_n=\begin{cases}1 & \text{w.p. } p_n\\ 0 & \text{w.p. } 1-p_n.\end{cases}$$

   Let $Y_n=\prod\limits_{k=1}^{n}X_k$. Workout explicit conditions on the sequence $\{p_n\}$ that ensure (a) $Y_n\stackrel{p}{\to}0$, or (b) $Y_n\stackrel{p}{\to}1$ or (c) for any $0\leq\alpha\leq 1, Y_n\stackrel{d}{\longrightarrow}Y$, where

$$Y=\begin{cases}1 & \text{w.p. } \alpha\\ 0 & \text{w.p. } 1-\alpha.\end{cases}$$

7. Let $c\in\mathbb{R}$. Let Y_n,Y be a sequence of random variables. Let g is a continuous function
   (a) If $Y_n\stackrel{a.e.}{\longrightarrow}Y$ then show that $g(Y_n)\stackrel{a.e.}{\longrightarrow}g(Y)$.
   (b) If $Y_n\stackrel{p}{\to}c$ then show that $g(Y_n)\stackrel{p}{\to}g(c)$.

8. Consider the t-distribution with n degrees of freedom. Show that this distribution converges weakly to the standard Normal distribution as $n\to\infty$.

9. Let X_n be a sequence of random variables. Let X be a degenerate random variable. Suppose $X_n\stackrel{d}{\longrightarrow}X$, then show that $X_n\stackrel{p}{\to}X$.

10. Let $X,\ X_n$ be a sequence of random variables. Let g be a bounded continuous function. Suppose $X_n\stackrel{d}{=}X$ then there exists a probability space $(\Omega,\mathcal{F},P)$ such that $\lim\limits_{n\to\infty}Eg(X_n)=Eg(X)$.

11. Let $X_n\stackrel{d}{=}B(n,p)$. Let $\hat{p}_n=\frac{X_n}{n}$. Let $g(t)=\sin^{-1}(\sqrt{t}), 0\leq t\leq 1$. Compute the assymptotic distribution of $\sqrt{n}[g(\hat{p}_n)-g(p)]$.

12. Let X_n be a sequence of $i.i.d.$ Poisson (θ) random variables and $T_n=\frac{X_1+X_2+\ldots X_n}{n}$. Define $g(t)=\sqrt{t}, t\geq 0$. Then compute the assymptotic distribution as $\sqrt{N}(g(T_n)-g(\theta))$.

Problems to be turned in are: 1,4,5,8,11