Due : Thursday, November 6th 2003

1. Let the one-step transition matrix P for a two state $S = \{1, 2\}$ discret-time Markov chain be given by
$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$
where $a, b \in [0, 1]$ but $a + b > 0$.
   (a) Find the eigen values λ_1, λ_2 of P
   (b) Find matrices U_1, U_2 such that
$$P^n = \lambda_1^n U_1 + \lambda_2^n U_2, \quad n = 0, 1, 2, 3, \ldots$$
   (c) Show that there exists a matrix Π such that $\Pi := \lim_{n\to\infty} P^n$ (i.e. limit component by component). Give a probabilistic interpretation for Π.
   (d) Let $a = \frac{1}{3}$ and $b = \frac{1}{2}$. Find $P(X_2 = 2, X_6 = 1 \mid X_0 = 1)$.

2. There are two light switches for two lights. Initially, all of the lights are off. At times $1, 2, 3, \ldots$ a switch is chosen at random and is then flicked (changing it from Off to On, or from On to Off).
   (a) Write down the state space and transition matrix of the Markov Chain describing this process.
   (b) How long does it take on average until all lights are on ?
   (c) What is the probability that light two was on before light one is switched on for the first time ?

3. Consider the following random walk on the state space $S = \{-3, -2, -1, 0, 1\}$. Let its transition matrix be given by:
$$P = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$
   (a) Decompose the state space into set of transient states and recurrent classes.
   (b) For each recurrent class R, let Y be X restricted[1] to R.
      i. Write down a one-step transition matrix P for Y.
      ii. Find the stationary disribution of Y.
      iii. Show that π does not satisfy the detailed balance condition.
      iv. Find $E(T_i \mid X_0 = i)$ for each $i \in R$
      v. Discuss whether each state in Y is positive or null recurrent.

4. Roll a fair die repeatedly and let $Y_1, Y_2, \ldots$ be the resulting numbers. Let $X_n = \mid \{Y_1, Y_2, \ldots, Y_n\} \mid$ be the number of values we have seen in the first n rolls for $n \geq 1$ and set $X_0 = 0$.

[1] Since R is closed. If X starts in R it stays in R. So we can think X with a restricted state space R instead of the bigger S. We call this chain as Y.

(a) Find the transition probability matrix.

(b) Let $T = \min\{n \geq 0 : X_n = 6\}$ be the number of trials we need to see all 6 numbers atleast once. Find $E(T)$.

5. Consider the points $1, 2, 3, 4$ to be marked on a straight line. Let X_n be a Markov Chain that moves to the right with probability $\frac{2}{3}$ and to the left with probability $\frac{1}{3}$, but subject this to the rule that if X_n tries to go left from 1 or to the right from 4 it stays put.

   (a) Find the transition probability matrix for the chain, and

   (b) Find the limiting fraction of time the chain spends at each site.

6. Supandi has three umbrellas, some at his office, and some at home. If he is leaving home in the morning (or leaving work at night) and it is raining, he will take an umbrella, if one is there. Otherwise, Supandi gets wet. Assume that independent of the past, it rains on each trip with probability 0.2. To formulate a Markov Chain, let X_n be the number of umbrellas at his current location.

   (a) Find the transition probability for this Markov Chain.

   (b) Calculate the limiting fraction of time he gets wet.

7. Consider the Markov chain with state space $S = \{0, 1, 2, \ldots\}$, with the transition matrix P such that $p(i, i-1) = 1$ forall $i > 0$ and $p(0, i) = p_i$ for all $i \geq 0$. Assume that $\sum_{i=0}^{\infty} p_i = 1$

   (a) Decide when the chain is recurrent.

   (b) Give necessary and sufficient conditions when the chain is positive recurrent.

8. Consider the Markov chain with state space $S = \{0, 1, 2, \ldots\}$, with the transition matrix P such that $p(i, i+1) = 1 - p_i$ forall $i > 0$ and $p(i, 0) = p_i$ for all $i \geq 0$, and $0 < p_i < 1$.

   (a) Find the conditions that gaurantee that 0 is recurrent.

   (b) Find the conditions that gaurantee that 0 is positive recurrent.

   (c) Find the stationary distribution.

9. Let $0 < p < 1$. Consider a markov chain X_n with state space $\{0, 1, 2, \ldots\}$ with the transition matrix P such that $p(i, i+1) = p, p(i, i-1) = 1 - p$, for $i \geq 1$ and $p(0, 1) = 1$.

   (a) State and solve the detailed balance equations for P.

   (b) Discuss when the stationary distribution exists by using the above solution.

   (c) Using the above identify conditions under which the chain is transient or recurrent or positive recurrent.

   (d) When the chain is recurrent or postive recurrent calculate $E(T_i \mid X_0 = i)$ and limiting fraction of visits to state i.

10. Consider a markov chain X_n with state space $S = \mathbb{Z}$ with the transition matrix P such that $p(i, i+1) = p, p(i, i-1) = 1 - p$, for $i \in \mathbb{Z}$.

   (a) Decide when the chain is recurrent and transient.

   (b) Decide whether the random walk has a stationary distribution or not.

Problems to be turned in:1,6,9.