Due : Tuesday, 11th August 2003

1. Find the characteristic function of the Gamma distribution with parameters (n, α). Hence or otherwise, derive the characteristic function for the chi-squared distribution with k degrees of freedom.

2. The sizes of particles used in sedimentation experiments often have uniform distributions. Suppose that spherical particles have diameters uniformly distributed between 0.01 and 0.05 centimeters. Find the mean and variance of the volumes of these particles.

3. Customers arrive at a store according to a Poisson process with rate 2. Let $N(t)$ be the number of customers that arrive from store opening (at time zero) to time t. Let $0 < x < y$.

   (a) Find the conditional distribution of $N(x)$ given that $N(y) = n$.

   (b) Find the conditional distribution of $N(y)$ given that $N(x) = n$.

   (c) Find the mean, variance and probability density function of the waiting time between the opening of the counter and the arrival of the second customer

4. The joint distribution of amount of pollutant emitted from a smokestack without a cleaning device (X_1) and with a cleaning device (X_2) is given by

$$f(x_1, x_2) = \begin{cases} k & \text{if } 0 \le x_1 \le 2, 0 \le x_2 \le 1 \text{ and } 2x_2 \le x_1 \\ 0 & \text{otherwise} \end{cases}$$

   The reduction in amount of pollutant emitted due to the cleaning device is given by $U = X_1 - X_2$.

   (a) Find the probability density function for U.

   (b) Find $E(U)$.

5. The length of time (in appropriate units) that a certain type of component functions before failing is a random variable with probability density function

$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Once the component fails it is immediately replaced with another one of the same type.

   (a) If we let X_i denote the lifetime of the i^{th} component to be put in use, then $S_n = \sum_{i=1}^{n} X_i$ represents the time of the n^{th} failure. The long–term rate at which failures occur is

$$r = \lim_{n \to \infty} \frac{n}{S_n}$$

   Determine r, assuming that the random variables X_i are independent.

   (b) How many components would one need to have on hand to be approximately 90% certain that the stock would last at least 35 units of time?

6. Two types of coin are produced at a factory: a fair coin and a biased one that comes up heads 55% of the time. We have one of these coins but do not know whether it is a fair or biased coin. In order to ascertain which type of coin we have, we shall perform the following statistical test. We shall toss the coin 1000 times. If the coin comes up heads 525 or more times we shall conclude that it is a biased coin. Otherwise, we shall conclude that it is fair. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

Problems to be turned in are: 3(c), 4(a), 5.