Question: Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of real numbers and let $\{z_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $z_n = x_n + y_n$ for all $n \in \mathbb{N}$. Show that

$$\liminf_{n\to\infty} z_n \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$$

provided that the sum on the right hand side is not of the form $\infty - \infty$.

Solution:

Let $X = \liminf_{n\to\infty} x_n, Y = \liminf_{n\to\infty} y_n, Z = \liminf_{n\to\infty} z_n$.

Case 1: Both X and Y are finite

For any $\epsilon > 0$, only a finite number of terms of $\{x_n\}_{n=1}^{\infty}$ are less than $X - \epsilon/2$; and only a finite number of terms of $\{y_n\}_{n=1}^{\infty}$ are less than $Y - \epsilon/2$. Suppose for some $k \in \mathbb{N}, x_k \geq X - \epsilon/2$, and $y_k \geq Y - \epsilon/2$. Then $z_k = x_k + y_k \geq X + Y - \epsilon$. Hence if $z_p < X + Y - \epsilon$, then $x_p < X - \epsilon/2$ or $y_p < Y - \epsilon/2$. As the number of p for which the second condition is satisfied is finite, the number of p for which $z_p < X + Y - \epsilon$ is also finite.

We will show by contradiction that $Z \geq X + Y$. Suppose $Z < X + Y$. Let $\epsilon = (X + Y - Z)/2$. As Z is a limit point of $\{z_n\}_{n=1}^{\infty}$, there exist infinitely many p for which $|z_p - Z| < \epsilon$. But $|z_p - Z| < \epsilon$ implies $z_p < X + Y - \epsilon$, and we know that there are only finitely many p for which $z_p < X + Y - \epsilon$. This is a contradiction, and hence $Z \geq X + Y$.

Case 2: At least one of X and Y is non-finite

Assume without loss of generality that X is non-finite. It is given that $\{X, Y\} \neq \{\infty, -\infty\}$. If $X = -\infty$, then $X + Y = -\infty$. In this case, clearly $Z \geq X + Y$. Now consider the case when $X = \infty$. As $Y \neq -\infty$, the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded below by, say, M. For, if it did not have a lower bound, it would have a subsequence converging to $-\infty$, which contradicts the fact that $Y \neq -\infty$.

As $\limsup_{n\to\infty} x_n \geq X, \limsup_{n\to\infty} x_n = \infty = X$. Hence $\{x_n\}_{n=1}^{\infty}$ converges to ∞. For all $P \in \mathbb{R}$, there exists a $N_0 \in \mathbb{N}$ such that $x_n > P - M$ for all $n \geq n_0$. $x_n > P - M$ implies $x_n + M > P$, which implies $x_n + y_n > P$. Hence for all $P \in \mathbb{R}$, there exists a $N_0 \in \mathbb{N}$ such that $z_n > P$ for all $n \geq n_0$. This shows that $\{z_n\}_{n=1}^{\infty}$ converges to ∞ and so $Z = \infty$. Hence $Z \geq \infty = X + Y$.