Due: April, 4th 2003.

1. Let $B(t)$ be a standard Brownian motion starting at 0 on $(\Omega, \mathcal{F}, P)$. Let $X(t)$ be a solution of the stochastic differential equation:

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

   (a) Let $b(x) = bx$ and $\sigma(x) = \sigma x$. Derive an expression the transition density $p(s,t;x,y)$ for the above proces and verify that it satisfies kolmogorov backward equation:

$$p_s + bxp_x + \frac{1}{2}\sigma^2 x^2 p_x x = 0$$

   (b) Let b and σ be lipschitz continuous functions. Let $0 \leq t \leq T$ and $v(t,x) = E^{t,x}(h(X(T))$. Assume that X has a transition density $p(s,t;x,y)$. Using the fact that $v(t,x)$ satisfies the Kolmogrov backward equation, show that p also does.

2. Let $B(t)$ be a standard Brownian motion starting at 0 on $(\Omega, \mathcal{F}, P)$. Let $T_b = \inf\{t : B_t = b\}$.

   (a) (Reflection principle) Now argue heurisitcally

$$P(T_b < t \cap B_t < b) = P(T_b < t \cap B_t > b).$$

   How would you make this argument rigourous ?

   (b) Suppose $f_b(y)$ was the density of the random variable T_b show that

$$f_b(t) = \begin{cases} 0 & t \leq 0 \\ \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} & t > 0 \end{cases}$$