Due: March 28, 2003.

Let $B(t)$ be a Brownian motion adapted to the filtration $\mathcal{F}_t$. Let δ_t be an adaptive process.

1. Let δ be an elementary process and let $I(t) = \int_0^t \delta_s dB(s)$ be the ito integral as defined in class. Show that
   (a) $I(t)$ is a martingale with respect to the filtration $\mathcal{F}_t$.
   (b) $E(I(t)^2) = E\int_0^t \delta_u^2 B(u)$.
   (c) The Quadratic variation of $I(t) = \int_0^t \delta_u^2 dB(u)$

2. Let the stock price process $S(t)$ be modeled as solution of the following stochastic differential equation:
$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{1}$$
   (a) Let $v(t,x)$ be a function of two variables in time and space. Assume that the partial derivatives v_t, v_x, v_{xx} all exist. Write out an integral equation that $v(t, S(t))$ satisfies.
   (b) Calculate $E(S(t))$.
   (c) Using Ito's formula calculate $E(S(t)^2)$.

3. Let S_n be a simple random walk on $\mathbf{Z}$. Let $n > 0$ and z be an integers.
   (a) A path from the origin to the point (n,z) is a polygonal line whose vertices have abscissas (first-coordinate) $0, 1, \ldots, n$ and ordinates (second-coordinate) $0, s_1, s_2, \ldots, s_n$ satisfying $s_k - s_{k-1} = x_k = \overset{+}{-} 1, k = 1, ..., n$ with $s_n = z$ n will be called the length of the path. Let p among the x_k be positive and q among them be negative.
      i. Write an expression for n and z in terms of p and q ?
      ii. Let $N_{n,z}$ denote the number of different paths from the origin to an arbitrary point (n,z). Write an expression for n and z in terms of p and q.
   (b) (Reflection principle) Let x, n be positive integers and p be any real number. Show that the number of paths from $A(0,x)$ to some point $B(n,p)$ which touch or cross the $x - axis$ equals the number of all paths from $A'(0,-x)$ to B.
   (c) Let n, x be positive integers. There are exactly $\frac{xN_{n,x}}{n}$ paths $(0, s_1, s_2, \ldots, s_n = x)$ from the origin to the point (n,x) such that $s_1 > 0, \ldots, s_n > 0$.
   (d) (Ballot Theorem) Suppose that, in a ballot, candidate P scores p votes and candidate Q scores q votes, where $p > q$. The probability that throughout the counting there are always more votes for P than for Q equals $\frac{p-q}{p+q}$.