Due: January 24th.

Key words-
Mathematics: $\sigma-$algebra, generators, random variable, conditional expectation, martingales.
Finance: Simple European Derivative Security, Hedgeable, Portfolio process, Self-financing value, Arbitrage pricing theory (APT) value.

1. Define what is meant by the generator(s) of a σ-algebra.

2. Consider the Binomial model with $n = 4$ as defined in class (not in the previous assignment). Let Ω_4 and $\mathcal{F} = \mathcal{P}(\Omega_4)$. Describe $\mathcal{F}_1$ and its generators. Describe $\mathcal{F}_3$.

3. Cards are drawn from an ordinary deck of 52, one at a time, randomly and with replacement. Let X and Y denote the number of draws until the first ace and first king are drawn, respectively. Find $E(X|Y = 5)$.

4. Suppose that X and Y are random variables with joint probability density

$$f(x, y) = \begin{cases} \frac{4}{5}(xy + 1) & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are X and Y independent?
   (b) Compute the marginal densities $f_X(x)$, $f_Y(y)$ and the conditional density $f_{X|Y}(x|y)$.
   (c) Calculate the means μ_X, μ_Y, the variances σ^2_X, σ^2_Y and the covariance σ_{XY}.
   (d) Calculate $E(X^2 + Y^2)$.

5. Consider the Binomial model with $n = 3$ as defined in class (not in the previous assignment). Let the probability of getting a head in a coin toss be $\frac{1}{4}$. Let $u = 4$ and $d = 0.25$ Show that $E(S_3 \mid \mathcal{F}_2)(\omega) = 1.33 S_2(\omega), \quad \forall \omega \in \Omega_3$

6. Consider the Binomial model with generic n as defined in class (not in the previous assignment). Show that the self-financing value of the portfolio process X_k is a martingale with respect to $\mathcal{F}_k$ and conclude that it satisfies the following property

$$\tilde{E}((1 + r)^{-k} X_k \mid \mathcal{F}_l) = (1 + r)^{-l} X_l,$$

   where $l < k < n$ and $\tilde{E}$ is the expectation under the risk neutral measure.