Due: January 17th.

Please read chapter 2 of Shreve's notes before beginning this assignment. You may not know some of the technical terms at these times pleae refer to chapter 1. Here are some of the keys to achieve while reading.

1. Doing calculations like in 2.3.1
2. Learning how to use the properties illustrated in Section 2.3.4.
3. Doing calculations like in 2.3.5

Lets try and do the calculations in 2.3.1 and 2.3.5 in a mildly different way. See if you like this better.

Let $p, q > 0$ such that $p+q = 1$. Let $r > 0$ be given. Let $d < 1+r < u$. Let $\Omega_i = \{H, T\}$ and $\Omega = \prod_{i=1}^{n} \Omega_i$. Let P be probability on Ω, such that $P(\omega_1, \ldots, \omega_n) = p^{\#\{j:\omega_j=H\}} q^{\#\{j:\omega_j=T\}}$. Let E denote the expectation under P. Define the random variables $\xi_i : \Omega_i \to \{u, d\}$, such that $\xi_i(H) = u$ and $\xi_i(T) = d$. Let S_0 be the stock price at time 0. We can then define

$$S_k(\omega) = S_k(\omega_1, \ldots \omega_k) = \left(\prod_{i=1}^{k} \xi_i(\omega_i) \right) S_0$$

We have done nothing fancy, just specified notation to denote the value of the stock price at time k

Conditional Expectation

1. Using the above definition solve the following questions:
   (a) Assume $n = 4$. Calculate $S_2(\{H, H, T, H\})$ and $S_4(\{H, H, T, H\})$.
   (b) Show by calculating above definitions that $S_3(\{H, H, H, H\}) = S_3(\{H, H, H, T\})$. Give an intuitive explanation why this is true.
   (c) Assume $n = 3$. Sketch the binomial tree of possibilities for S_k from the above definition. Check that the tree is the same as on page 50.
2. Assume $n = 4$.

(a) Show that $\xi_1, \xi_2, \xi_3, \xi_4$ are all independent random variables. Calculate $E(\xi_1)$.

(b) Use Property (j) on page 56 to show that $E(S_3 \mid \mathcal{F}_2)(\omega) = S_2 E(\xi_3 \mid \mathcal{F}_2)(\omega)$

(c) Using 2 (a), 2(b) and Property (k) on page 56 conclude that

$$E(S_3 \mid \mathcal{F}_2) = (pu + qd)S_2$$

(d) Using the above ideas or Property (i) on page 56 conclude that

$$E(S_4 \mid \mathcal{F}_1) = (pu + qd)S_1$$

(e) $\mathcal{F}_0$ i.e information generated by S_0 contains no information about the future. So convince yourself via Property (k) on page 56 that

$$E(S_k \mid \mathcal{F}_0) = E(S_k), k = 1, 2, 3, 4$$

Martingales

Please read the definition on page 58. Let $\tilde{\mathbb{P}}$ be the risk neutral probability defined in class on Ω and $\tilde{E}$ denote the corresponding expectation. Most of these problems are worked out in chapter 3.

1. Use the notation described above. Assume $n = 4$. Show that under $\tilde{P}$, $\{(1+r)^{-k}S_k, \mathcal{F}_k\}$ is a martingale. I.e.

$$\tilde{E}((1+r)^{-(k+1)}S_{k+1} \mid \mathcal{F}_k) = (1+r)^{-k}S_k$$

You may refer to page 62 where this calculation is done. But justification for the first equality in the proof is not given. So, please do the honours.

2. Use the notation described above. Assume $n = 4$. Show that under $\tilde{P}$, $\{(1+r)^{-k}X_k, \mathcal{F}_k\}$ is a martingale. I.e.

$$\tilde{E}((1+r)^{-(k+1)}X_{k+1} \mid \mathcal{F}_k) = (1+r)^{-k}X_k$$

You may refer to page 63 where this calculation is done.

3. Under $\tilde{P}$ can you give an example of a process that is not a martingale.