Problems due:3, 6
Due date: July 29th, 2010

1. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that $C([a, b])$ equipped with the metric via the sup-norm is a complete metric space.

2. Let (S, d) be a metric space and T be a contraction from S into S. Show that T^k is a contraction for all $k \geq 1$ and each is uniformly continuous.

3. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $x : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that $x(0) = a \in \mathbb{R}$. Show that
$$x(t) = a + \int_0^t f(s, x(s))ds$$
if and only if x is differentiable in $(0, \infty)$ and
$$\frac{d}{dt}x(t) = f(t, x(t)),\ \forall\ t > 0.$$

4. Show that any Lipschitz function on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$ and has linear growth. Conversely, suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable such that f' is bounded, show that f is Lipschitz continuous.

5. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \sqrt{x}$. Show that f is not a Lipschitz continuous function on $[0, \infty)$.

6. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. Consider the initial value problem
$$\frac{d}{dt}x(t) = f(t, x(t)),\ \forall\ t > 0 \text{ with } x(0) = a \in \mathbb{R}.$$
   (a) Suppose $f(t, x) \equiv g(t)$ for some continuous g. Is it possible that the initial value problem does not have a unique solution ?
   (b) Suppose f is Lipschitz continuous in x-variable but not uniformly in t. What can you say about the solution set to the initial value problem ?
   (c) Can you give an example of a continuous f such that the initial value problem does not have a solution ?

7. Let (S, d) be a metric space and T be a map from S into S. Assume further that $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in S$. Does it necessarily imply that T has a fixed point ? Suppose T has a fixed point, does it imply that it is unique.