1. Consider $y = xe^{-x}$, for $0 \leq x \leq 8$. Write a function file, using `hermint` , that creates a piecewise-cubic Hermite approximations with $4, 6, 8, 12$ equally spaced points. Plot all 4 of these curves and the function on 4 different graphs.
2. Find the cubic-spline passing through $(x, y) = (1,1), (2,3), (3,2)$ and $(4,4)$. and having zero slope at $x = 1$ and $x = 4$ using `splintFE`. Plot the spline.
3. Show the following Theorem for $n = 2$ case.

   Theorem:*Assume that $f \in C^n([a,b])$ and that $x_1, x_2, \ldots x_n \in [a,b]$ are n nodes. If $x \in [a,b]$, then*

   $$f(x) = P_{n-1}(x) + e_{n-1}(x),$$

   where P_{n-1} is the Lagrange Polynomial of order $n-1$ and

   $$e_{n-1}(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)f^n(c)}{n!}$$

   for some value $c \equiv c(x)$.

Bonus Questions [1]

Without quoting theorems and results please solve the following.

1. Let $(x_i, f(x_i), f'(x_i)), i = 1, \ldots n$ be given. Let

   $$P_i(x) = a_i + b_i(x - x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3,$$

   be the cubic interpolant in the range $[x_i, x_{i+1}]$. Show that under the following constraints:

   $$P_i(x_i) = f(x_i), P_i'(x_i) = f'(x_i), P_i(x_{i+1}) = f(x_{i+1}), P_i'(x_{i+1}) = f'(x_{i+1}), 1 \leq i \leq n-1,$$

   $$\begin{aligned} a_i &= f(x_i), \\ b_i &= f'(x_i), \\ c_i &= \frac{3f[x_i, x_{i+1}] - 2f'(x_i) - f'(x_{i+1})}{(x_{i+1} - x_i)} \\ d_i &= \frac{f'(x_i) - 2f[x_i, x_{i+1}] + f'(x_{i+1})}{(x_{i+1} - x_i)^2} \end{aligned}$$

2. Complete the proof of cubic splines outlined in class for all the three possibilities.

[1] The (best) complete answer will get half a kilogram of mangoes and a loaf of Vanilla icecream.