Chapter 3

Arbitrage Pricing

3.1 Binomial Pricing

Return to the binomial pricing model

Please see:

- Cox, Ross and Rubinstein, *J. Financial Economics*, **7**(1979), 229–263, and
- Cox and Rubinstein (1985), **Options Markets**, Prentice-Hall.

Example 3.1 (Pricing a Call Option) Suppose $u = 2, d = 0.5, r = 25\%$(interest rate), $S_0 = 50$. (In this and all examples, the interest rate quoted is per unit time, and the stock prices $S_0, S_1, \ldots$ are indexed by the same time periods). We know that

$$S_1(\omega) = \begin{cases} 100 & \text{if } \omega_1 = H \\ 25 & \text{if } \omega_1 = T \end{cases}$$

Find the value *at time zero* of a call option to buy one share of stock at time 1 for \$50 (i.e. the *strike price* is \$50).

The value of the call at time 1 is

$$V_1(\omega) = (S_1(\omega) - 50)^+ = \begin{cases} 50 & \text{if } \omega_1 = H \\ 0 & \text{if } \omega_1 = T \end{cases}$$

Suppose the option sells for \$20 at time 0. Let us construct a portfolio:

1. Sell 3 options for \$20 each. Cash outlay is $-\$60$.
2. Buy 2 shares of stock for \$50 each. Cash outlay is \$100.
3. Borrow \$40. Cash outlay is $-\$40$.

This portfolio thus requires no initial investment. For this portfolio, the cash outlay at time 1 is:

	$\omega_1 = H$	$\omega_1 = T$
Pay off option	$\$150$	$\$0$
Sell stock	$-\$200$	$-\$50$
Pay off debt	$\$50$	$\$50$
	$-----$	$-----$
	$\$0$	$\$0$

The *arbitrage pricing theory (APT)* value of the option at time 0 is $V_0 = 20$. ∎

Assumptions underlying APT:

- Unlimited short selling of stock.
- Unlimited borrowing.
- No transaction costs.
- Agent is a "small investor", i.e., his/her trading does not move the market.

Important Observation: The APT value of the option does not depend on the probabilities of H and T.

3.2 General one-step APT

Suppose a derivative security pays off the amount V_1 at time 1, where V_1 is an $\mathcal{F}_1$-measurable random variable. (This measurability condition is important; this is why it does not make sense to use some stock unrelated to the derivative security in valuing it, at least in the straightforward method described below).

- Sell the security for V_0 at time 0. (V_0 is to be determined later).
- Buy Δ_0 shares of stock at time 0. (Δ_0 is also to be determined later)
- Invest $V_0 - \Delta_0 S_0$ in the money market, at risk-free interest rate r. ($V_0 - \Delta_0 S_0$ might be negative).
- Then wealth at time 1 is

$$\begin{aligned} X_1 &\triangleq \Delta_0 S_1 + (1+r)(V_0 - \Delta_0 S_0) \\ &= (1+r)V_0 + \Delta_0(S_1 - (1+r)S_0). \end{aligned}$$

- We want to choose V_0 and Δ_0 so that

$$X_1 = V_1$$

  regardless of whether the stock goes up or down.

The last condition above can be expressed by *two* equations (which is fortunate since there are *two* unknowns):

$$(1+r)V_0 + \Delta_0(S_1(H) - (1+r)S_0) = V_1(H) \tag{2.1}$$

$$(1+r)V_0 + \Delta_0(S_1(T) - (1+r)S_0) = V_1(T) \tag{2.2}$$

Note that this is where we use the fact that the derivative security value V_k is a function of S_k, i.e., when S_k is known for a given ω, V_k is known (and therefore non-random) at that ω as well. Subtracting the second equation above from the first gives

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{2.3}$$

Plug the formula (2.3) for Δ_0 into (2.1):

$$\begin{aligned}
(1+r)V_0 &= V_1(H) - \Delta_0(S_1(H) - (1+r)S_0) \\
&= V_1(H) - \frac{V_1(H) - V_1(T)}{(u-d)S_0}(u-1-r)S_0 \\
&= \frac{1}{u-d}[(u-d)V_1(H) - (V_1(H) - V_1(T))(u-1-r)] \\
&= \frac{1+r-d}{u-d}V_1(H) + \frac{u-1-r}{u-d}V_1(T).
\end{aligned}$$

We have already assumed $u > d > 0$. We now also assume $d \leq 1+r \leq u$ (otherwise there would be an arbitrage opportunity). Define

$$\tilde{p} \triangleq \frac{1+r-d}{u-d}, \quad \tilde{q} \triangleq \frac{u-1-r}{u-d}.$$

Then $\tilde{p} > 0$ and $\tilde{q} > 0$. Since $\tilde{p} + \tilde{q} = 1$, we have $0 < \tilde{p} < 1$ and $\tilde{q} = 1 - \tilde{p}$. Thus, $\tilde{p}, \tilde{q}$ are like probabilities. We will return to this later. Thus the price of the call at time 0 is given by

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]. \tag{2.4}$$

3.3 Risk-Neutral Probability Measure

Let Ω be the set of possible outcomes from n coin tosses. Construct a probability measure $\widetilde{I\!P}$ on Ω by the formula

$$\widetilde{I\!P}(\omega_1, \omega_2, \ldots, \omega_n) \triangleq \tilde{p}^{\#\{j;\omega_j = H\}}\tilde{q}^{\#\{j;\omega_j = T\}}$$

$\widetilde{I\!P}$ is called the *risk-neutral probability measure*. We denote by $\widetilde{I\!E}$ the expectation under $\widetilde{I\!P}$. Equation 2.4 says

$$V_0 = \widetilde{I\!E}\left(\frac{1}{1+r}V_1\right).$$

Theorem 3.11 *Under $\widetilde{I\!P}$, the discounted stock price process $\{(1+r)^{-k}S_k, \mathcal{F}_k\}_{k=0}^n$ is a martingale.*

Proof:

$$\begin{aligned}
&\widetilde{I\!E}[(1+r)^{-(k+1)}S_{k+1}|\mathcal{F}_k] \\
&= (1+r)^{-(k+1)}(\tilde{p}u+\tilde{q}d)S_k \\
&= (1+r)^{-(k+1)}\left(\frac{u(1+r-d)}{u-d}+\frac{d(u-1-r)}{u-d}\right)S_k \\
&= (1+r)^{-(k+1)}\frac{u+ur-ud+du-d-dr}{u-d}S_k \\
&= (1+r)^{-(k+1)}\frac{(u-d)(1+r)}{u-d}S_k \\
&= (1+r)^{-k}S_k.
\end{aligned}$$

3.3.1 Portfolio Process

The portfolio process is $\Delta = (\Delta_0, \Delta_1, \ldots, \Delta_{n-1})$, where

- Δ_k is the number of shares of stock held between times k and $k+1$.
- Each Δ_k is $\mathcal{F}_k$-measurable. (No insider trading).

3.3.2 Self-financing Value of a Portfolio Process Δ

- Start with nonrandom initial wealth X_0, which need not be 0.
- Define recursively

$$\begin{aligned}
X_{k+1} &= \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k) &\quad (3.1)\\
&= (1+r)X_k + \Delta_k(S_{k+1} - (1+r)S_k). &\quad (3.2)
\end{aligned}$$

- Then each X_k is $\mathcal{F}_k$-measurable.

Theorem 3.12 *Under $\widetilde{I\!P}$, the discounted self-financing portfolio process value $\{(1+r)^{-k}X_k, \mathcal{F}_k\}_{k=0}^n$ is a martingale.*

Proof: We have

$$(1+r)^{-(k+1)}X_{k+1} = (1+r)^{-k}X_k + \Delta_k\left((1+r)^{-(k+1)}S_{k+1} - (1+r)^{-k}S_k\right).$$

Therefore,

$$\begin{aligned}
&\widetilde{I\!\!E}[(1+r)^{-(k+1)}X_{k+1}|\mathcal{F}_k] \\
&\quad = \widetilde{I\!\!E}[(1+r)^{-k}X_k|\mathcal{F}_k] \\
&\qquad +\widetilde{I\!\!E}[(1+r)^{-(k+1)}\Delta_k S_{k+1}|\mathcal{F}_k] \\
&\qquad -\widetilde{I\!\!E}[(1+r)^{-k}\Delta_k S_k|\mathcal{F}_k] \\
&\quad = (1+r)^{-k}X_k \quad \text{(requirement (b) of conditional exp.)} \\
&\qquad +\Delta_k\widetilde{I\!\!E}[(1+r)^{-(k+1)}S_{k+1}|\mathcal{F}_k] \quad \text{(taking out what is known)} \\
&\qquad -(1+r)^{-k}\Delta_k S_k \quad \text{(property (b))} \\
&\quad = (1+r)^{-k}X_k \quad \text{(Theorem 3.11)}
\end{aligned}$$

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3.4 Simple European Derivative Securities

Definition 3.1 () A *simple European derivative security* with expiration time m is an $\mathcal{F}_m$-measurable random variable V_m. (Here, m is less than or equal to n, the number of periods/coin-tosses in the model).

Definition 3.2 () A simple European derivative security V_m is said to be *hedgeable* if there exists a constant X_0 and a portfolio process $\Delta = (\Delta_0, \ldots, \Delta_{m-1})$ such that the self-financing value process $X_0, X_1, \ldots, X_m$ given by (3.2) satisfies

$$X_m(\omega) = V_m(\omega), \quad \forall \omega \in \Omega.$$

In this case, for $k = 0, 1, \ldots, m$, we call X_k the *APT value at time k of V_m.*

Theorem 4.13 (Corollary to Theorem 3.12) *If a simple European security V_m is hedgeable, then for each $k = 0, 1, \ldots, m$, the APT value at time k of V_m is*

$$V_k \stackrel{\triangle}{=} (1+r)^k\widetilde{I\!\!E}[(1+r)^{-m}V_m|\mathcal{F}_k]. \tag{4.1}$$

Proof: We first observe that if $\{M_k, \quad \mathcal{F}_k; k = 0, 1, \ldots, m\}$ is a martingale, i.e., satisfies the martingale property

$$\widetilde{I\!\!E}[M_{k+1}|\mathcal{F}_k] = M_k$$

for each $k = 0, 1, \ldots, m-1$, then we also have

$$\widetilde{I\!\!E}[M_m|\mathcal{F}_k] = M_k, k = 0, 1, \ldots, m-1. \tag{4.2}$$

When $k = m-1$, the equation (4.2) follows directly from the martingale property. For $k = m-2$, we use the tower property to write

$$\begin{aligned}
\widetilde{I\!\!E}[M_m|\mathcal{F}_{m-2}] &= \widetilde{I\!\!E}[\widetilde{I\!\!E}[M_m|\mathcal{F}_{m-1}]|\mathcal{F}_{m-2}] \\
&= \widetilde{I\!\!E}[M_{m-1}|\mathcal{F}_{m-2}] \\
&= M_{m-2}.
\end{aligned}$$

We can continue by induction to obtain (4.2).

If the simple European security V_m is hedgeable, then there is a portfolio process whose self-financing value process $X_0, X_1, \ldots, X_m$ satisfies $X_m = V_m$. By definition, X_k is the APT value at time k of V_m. Theorem 3.12 says that

$$X_0, (1+r)^{-1}X_1, \ldots, (1+r)^{-m}X_m$$

is a martingale, and so for each k,

$$(1+r)^{-k}X_k = \widetilde{I\!\!E}[(1+r)^{-m}X_m|\mathcal{F}_k] = \widetilde{I\!\!E}[(1+r)^{-m}V_m|\mathcal{F}_k].$$

Therefore,

$$X_k = (1+r)^k\widetilde{I\!\!E}[(1+r)^{-m}V_m|\mathcal{F}_k].$$

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3.5 The Binomial Model is Complete

Can a simple European derivative security always be hedged? It depends on the model. If the answer is "yes", the model is said to be *complete.* If the answer is "no", the model is called *incomplete.*

Theorem 5.14 *The binomial model is complete. In particular, let V_m be a simple European derivative security, and set*

$$V_k(\omega_1, \ldots, \omega_k) = (1+r)^k\widetilde{I\!\!E}[(1+r)^{-m}V_m|\mathcal{F}_k](\omega_1, \ldots, \omega_k), \tag{5.1}$$

$$\Delta_k(\omega_1, \ldots, \omega_k) = \frac{V_{k+1}(\omega_1, \ldots, \omega_k, H) - V_{k+1}(\omega_1, \ldots, \omega_k, T)}{S_{k+1}(\omega_1, \ldots, \omega_k, H) - S_{k+1}(\omega_1, \ldots, \omega_k, T)}. \tag{5.2}$$

Starting with initial wealth $V_0 = \widetilde{I\!\!E}[(1+r)^{-m}V_m]$, the self-financing value of the portfolio process $\Delta_0, \Delta_1, \ldots, \Delta_{m-1}$ is the process $V_0, V_1, \ldots, V_m$.

Proof: Let $V_0, \ldots, V_{m-1}$ and $\Delta_0, \ldots, \Delta_{m-1}$ be defined by (5.1) and (5.2). Set $X_0 = V_0$ and define the self-financing value of the portfolio process $\Delta_0, \ldots, \Delta_{m-1}$ by the recursive formula 3.2:

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k).$$

We need to show that

$$X_k = V_k, \quad \forall k \in \{0, 1, \ldots, m\}. \tag{5.3}$$

We proceed by induction. For $k = 0$, (5.3) holds by definition of X_0. Assume that (5.3) holds for some value of k, i.e., for each fixed $(\omega_1, \ldots, \omega_k)$, we have

$$X_k(\omega_1, \ldots, \omega_k) = V_k(\omega_1, \ldots, \omega_k).$$

We need to show that

$$X_{k+1}(\omega_1,\ldots,\omega_k,H) = V_{k+1}(\omega_1,\ldots,\omega_k,H),$$

$$X_{k+1}(\omega_1,\ldots,\omega_k,T) = V_{k+1}(\omega_1,\ldots,\omega_k,T).$$

We prove the first equality; the second can be shown similarly. Note first that

$$\begin{aligned}
\widetilde{I\!\!E}[(1+r)^{-(k+1)}V_{k+1}|\mathcal{F}_k] &= \widetilde{I\!\!E}[\widetilde{I\!\!E}[(1+r)^{-m}V_m|\mathcal{F}_{k+1}]|\mathcal{F}_k] \\
&= \widetilde{I\!\!E}[(1+r)^{-m}V_m|\mathcal{F}_k] \\
&= (1+r)^{-k}V_k
\end{aligned}$$

In other words, $\{(1+r)^{-k}V_k\}_{k=0}^n$ is a martingale under $\widetilde{I\!\!P}$. In particular,

$$\begin{aligned}
V_k(\omega_1,\ldots,\omega_k) &= \widetilde{I\!\!E}[(1+r)^{-1}V_{k+1}|\mathcal{F}_k](\omega_1,\ldots,\omega_k) \\
&= \frac{1}{1+r}\left(\tilde{p}V_{k+1}(\omega_1,\ldots,\omega_k,H) + \tilde{q}V_{k+1}(\omega_1,\ldots,\omega_k,T)\right).
\end{aligned}$$

Since $(\omega_1,\ldots,\omega_k)$ will be fixed for the rest of the proof, we simplify notation by suppressing these symbols. For example, we write the last equation as

$$V_k = \frac{1}{1+r}\left(\tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T)\right).$$

We compute

$$\begin{aligned}
&X_{k+1}(H) \\
&= \Delta_k S_{k+1}(H) + (1+r)(X_k - \Delta_k S_k) \\
&= \Delta_k\left(S_{k+1}(H) - (1+r)S_k\right) + (1+r)V_k \\
&= \frac{V_{k+1}(H)-V_{k+1}(T)}{S_{k+1}(H)-S_{k+1}(T)}\left(S_{k+1}(H) - (1+r)S_k\right) \\
&\quad + \tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= \frac{V_{k+1}(H)-V_{k+1}(T)}{uS_k - dS_k}\left(uS_k - (1+r)S_k\right) \\
&\quad + \tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= \left(V_{k+1}(H) - V_{k+1}(T)\right)\left(\frac{u-1-r}{u-d}\right) + \tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= \left(V_{k+1}(H) - V_{k+1}(T)\right)\tilde{q} + \tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= V_{k+1}(H).
\end{aligned}$$

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