Due: Wednesday, September 5th, 2012
Problems to be turned in: 2,5

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers $\mathbb{R}$ and suppose that $x_n \to x$. Let $m, l \in \mathbb{N}$, $p : \mathbb{R} \to \mathbb{R}$ such that $p(x) = \sum_{k=0}^{l} p_k x^k$, and $q : \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $q(x) = \sum_{k=0}^{m} q_k x^k$, with $p_k \in \mathbb{R}, q_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$. Show that if $r : \mathbb{R} \to \mathbb{R}$ defined by $r(x) = \frac{p(x)}{q(x)}$ then $r(x_n) \to r(x)$.

2. Let $\alpha \in \mathbb{R}, p > 0$. Consider $\{x_n\}_{n=1}^{\infty}$, such that $x_n = \frac{n^\alpha}{(1+p)^n}$ for $n \in \mathbb{N}$. Decide if $\{x_n\}_{n=1}^{\infty}$ converges or not.

3. Consider $\{x_n\}_{n=1}^{\infty}$, such that $x_n = \left(1 + \frac{1}{n}\right)^n$, for all $n \in \mathbb{N}$. Show that x_n is a monotonically increasing sequence and it converges to $x \in \mathbb{R}$.

4. Let $a > 0$ and choose $s_1 > \sqrt{a}$. Define $s_{n+1} := \frac{1}{2}(s_n + \frac{a}{s_n})$ for $n \in \mathbb{N}$. Show that s_n is monotonically decreasing and[1] $\lim_{n\to} s_n = \sqrt{a}$.

5. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of non-zero real numbers such that $L := \lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|}$ exists. If $L < 1$, then $z_n \to 0$. What happens if $L > 1$?

6. Let $0 \le r < 1$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_n = \sum_{k=1}^{n} r^k$ for all $n \in \mathbb{N}$. Show that x_n is a convergent sequence.

7. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of integers. Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy implies that it is eventually a constant sequence (i.e. $\exists c \in \mathbb{N}, m \in \mathbb{N}$ such that $x_n = c$ for all $m \ge n$.)

8. Decide which one of the following sequences are cauchy.
   i) $\sum_{k=1}^{n} \frac{1}{k}$, (ii) $\frac{(-1)^n}{n}$, (iii) $\sum_{k=1}^{n} \frac{1}{k^2}$ and, (iv) $(-1)^n$

[1]Extra credit: If $z_n = s_n - \sqrt{a}$ then show that $z_{n+1} < \frac{z_n^2}{2\sqrt{a}}$. Justify the statement: "this is a good algorithm for calculating square roots".