6. (*Logarithm function:-* $\ln(x)$) Let E be the function as defined in q5. Let $L : (0,\infty) \to \mathbb{R}$ such that

$$L(E(y)) = y, \ \forall\, y \in \mathbb{R}$$

(a) Show that L is well-defined and $L(uv) = L(u) + L(v)$, for all $u, v \in (0,\infty)$. ($L(x)$ is denoted by $\ln(x)$ for all $x > 0$)

(b) Show that L is a continuous monotonically increasing (strictly) function.

(c) Show that for any $\alpha \in \mathbb{R}$, $x \in [0,\infty)$, $x^\alpha = E(\alpha(\ln(x))) = e^{\alpha \ln(x)}$.

Solution by Prateek Karandikar:

(a) Let $x \in \text{Range}(E)$. Then there exists $y \in \mathbb{R}$ such that $E(y) = x$. As E is strictly increasing, if $y_1 \neq y$ then $E(y_1) \neq E(y)$. Now it remains to be shown that $\text{Range}(E) = (0,\infty)$. $E(x) > 0$ for all $x \in \mathbb{R}$. Therefore $\text{Range}(E) \subseteq (0,\infty)$. Let $a \in (0,\infty)$. If $a = 1$, then $E(0) = a$. If $a > 1$, then $E(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!} > 1 + a$. $E(0) = 1$ and $E(a) > 1 + a$. By the intermediate value theorem there exists $b \in (0, a)$ such that $E(b) = a$. If $a < 1$, then by the above there exists $c \in (0, \frac{1}{a})$ such that $E(c) = \frac{1}{a}$ which implies $E(-c) = a$. Thus $(0,\infty) \subseteq \text{Range}(E)$ and hence $\text{Range}(E) = (0,\infty)$. So for all $x \in (0,\infty)$ there exists a unique $y \in \mathbb{R}$ such that $x = E(y)$. Hence L, defined as $L(E(y)) = y$, is well-defined.

We will now show that for all $u, v \in (0,\infty)$, $L(uv) = L(u) + L(v)$. Let $s = L(u)$ and $t = L(v)$. Then, $u = E(s)$ and $v = E(t)$. $uv = E(s)E(t) = E(s+t)$. Hence $L(uv) = s + t = L(u) + L(v)$.

(b) Let $x, y \in (0,\infty)$ such that $x < y$. Let $u = L(x)$ and $v = L(y)$. Then, $x = E(u)$ and $y = E(v)$. As E is strictly increasing, if $u \geq v$ then $x = E(u) \geq E(y) = y$, which is not true as $x < y$. Therefore $u < v$, and hence L is strictly increasing.

We will now show that L is continuous. Let $a \in (0,\infty)$. Let $\epsilon > 0$ be given. Let $b = L(a)$. Let $\delta = \min\big\{E(b+\epsilon) - E(b), E(b) - E(b-\epsilon)\big\}$. As E is strictly increasing, $\delta > 0$. If $c \in (0,\infty)$ such that $|c - a| < \delta$, then $c \in \big(E(b-\epsilon), E(b+\epsilon)\big)$. As L is strictly increasing, $L(c) \in (b-\epsilon, b+\epsilon)$, i.e. $|L(c) - L(a)| < \epsilon$. Hence L is continuous.

(c) If $x \in (0,\infty)$ and $L(x) = s$, then $x = E(s)$ and hence $E(L(x)) = E(s) = x$.

$$e^{\alpha \ln(x)} = \left(e^{\ln(x)}\right)^\alpha = \big(E(L(x))\big)^\alpha = x^\alpha$$

7. Find the continuity points of $f : \mathbb{R} \to \mathbb{R}$, when f is given by

(a) $f(x) = \lfloor x \rfloor$ (i.e. greatest integer less than or equal to x)

(b) $f(x) = x\lfloor x \rfloor$

(c) $f(x) = x - \lfloor x \rfloor$

Solution by Prateek Karandikar:

(a) Let $x \in \mathbb{R}$. Consider the case when $x \notin \mathbb{Z}$. Let $n = \lfloor x \rfloor$. Then $n < x < n+1$. Let $\epsilon > 0$ be given. Let $\delta = \min\{x-n, n+1-x\}$. $\delta > 0$. If $|y-x| < \delta$, then $y \in (n, n+1)$. Hence $|f(x) - f(y)| = |n-n| = 0 < \epsilon$. Therefore f is continuous at all $x \in \mathbb{R} - \mathbb{Z}$. Now consider the case when $x \in \mathbb{Z}$. Then $f(x) = x$. Let $\epsilon = \frac{1}{2}$. Let $\delta > 0$ be given. Choose $y = x - \min\{\frac{\delta}{2}, \frac{1}{2}\}$. Then $f(y) = x - 1$. $|y - x| < \delta$ and $|f(y) - f(x)| = 1 > \epsilon$. Hence f is discontinuous at all $x \in \mathbb{Z}$.

(b) Let $x \in \mathbb{R}$. Consider the case when $x \neq 0$. Then $f(x) = x\lfloor x \rfloor$ and $\lfloor x \rfloor = f(x)\frac{1}{x}$. The functions $g : \mathbb{R} - \{0\} \to \mathbb{R},\ g(x) = x$ and $h : \mathbb{R} - \{0\} \to \mathbb{R},\ h(x) = \frac{1}{x}$ are both continuous at all $x \in \mathbb{R} - \{0\}$. If two functions from $A \subseteq \mathbb{R}$ to $\mathbb{R}$ are continuous at a point, their product is also continuous at that point. Therefore, for $x \in \mathbb{R} - \{0\}$, $f(x)$ is continuous at x if and only if $\lfloor x \rfloor$ is continuous at x. Now consider the case when $x = 0$. Let $\epsilon > 0$ be given. Choose $\delta = \min\{\frac{1}{2}, \epsilon\}$. Let $y \in \mathbb{R}$ such that $|y - x| = |y| < \delta$. For $y \in [0, \delta)$, $f(y) = 0$. For $y \in (-\delta, 0)$, $f(y) = (-1)y$. Therefore $|f(y) - f(x)| = |f(y)| < \epsilon$. Hence f is continuous at 0.

For any $x \in \mathbb{R}$, f is continuous at x iff $x \in \big((\mathbb{R} - \mathbb{Z}) \cup \{0\}\big)$.

(c) The sum and difference of two continuous functions from $\mathbb{R} \to \mathbb{R}$ is also continuous. $f(x) = x - \lfloor x \rfloor$ and $\lfloor x \rfloor = x - f(x)$. Hence $f(x)$ is continous at x if and only if $\lfloor x \rfloor$ is continuous at x, i.e. if and only if $x \in \mathbb{R} - \mathbb{Z}$.

5. Let A be a countable subset of $\mathbb{R}$. Consider $p : A \to [0,1]$ such that $\sum_{n=1}^{\infty} p(x_n) = 1$ where $\{x_n\}_{n=1}^{\infty}$ is an enumeration of A. Define $F : \mathbb{R} \to [0,1]$ by

$$F(x) = \sum_{x_n \leq x} p(x_n) \equiv \sum_{n=1}^{\infty} g^x(x_n)p(x_n)$$

(a) Show that F is monotonically increasing.

(b) Identify the discontinuity points of F and show that $F(x+) = F(x)$ for all $x \in \mathbb{R}$.

(c) Show that $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

(d) By choosing a suitable A and p construct an example of a monotonically increasing function whose points of discontinuity are not isolated.

Solution by Prateek Karandikar:

For any $B \subseteq A$, $\sum_{x\in B} p(x)$ is well-defined as it is equal to $\sum_{n=1}^{\infty} 1_B(x_n)p(x_n)$ which by the comparison test converges absolutely.

(a) Let $p, q \in \mathbb{R}$ such that $p < q$. For all $x \in \mathbb{R}$, $g^p(x) \leq g^q(x)$. Therefore

$$F(p) = \sum_{n=1}^{\infty} g^p(x_n)p(x_n) \leq \sum_{n=1}^{\infty} g^q(x_n)p(x_n) = F(q)$$

Hence F is monotonically increasing.

(b) We will consider the right hand and left hand limits for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ be given. Let $\epsilon > 0$ be given. There exists $N_0 > 1 \in \mathbb{N}$ such that $\sum_{n=1}^{N_0} p(x_n) > 1 - \epsilon$, i.e.

$\sum_{n=N_0+1}^{\infty} p(x_n) < \epsilon$. Let $\delta = \min\left[\{|x-x_i| : 1 \leq i \leq N_0\} - \{0\}\right]$. $\delta > 0$. Let $y \in (x, x+\delta)$. Then, $x_j \in (x, y] \implies j > N_0$.

$$\begin{aligned} |F(y) - F(x)| &= \sum_{z \in (x,y] \cap A} p(z) (\text{why?}) \\ &\leq \sum_{n=N_0+1}^{\infty} p(x_n) \\ &< \epsilon \end{aligned}$$

For all $x \in \mathbb{R}$, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $y \in (x, x+\delta) \implies |F(y) - F(x)| < \epsilon$. Therefore $F(x+) = F(x)$ for all $x \in \mathbb{R}$. Now let $y' \in (x-\delta, x)$. $x_j \in (y', x) \implies j > N_0$. Define $G(x) = \sum_{x_n < x} p(x_n)$.

$$\begin{aligned} |F(y') - G(x)| &= \sum_{z \in (y',x) \cap A} p(z) \\ &\leq \sum_{n=N_0+1}^{\infty} p(x_n) \\ &< \epsilon \end{aligned}$$

For all $x \in \mathbb{R}$, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $y' \in (x-\delta, x) \implies |F(y') - G(x)| < \epsilon$. Therefore $F(x-) = G(x)$ for all $x \in \mathbb{R}$.

$F(x+) = F(x)$ for all $x \in \mathbb{R}$. Therefore,

$$\begin{aligned} F \text{ is not continuous at } x &\iff F(x-) \neq F(x) \\ &\iff G(x) \neq F(x) \\ &\iff \sum_{x_n < x} p(x_n) \neq \sum_{x_n \leq x} p(x_n) \\ &\iff x \in A \text{and} p(x) \neq 0 \end{aligned}$$

(c) Let $\epsilon > 0$ be given. There exists $N_0 \in \mathbb{N}$ such that $\sum_{n=1}^{N_0} p(x_n) > 1 - \epsilon$, i.e. $\sum_{n=N_0+1}^{\infty} p(x_n) < \epsilon$. Let $M = \max\{x_i : 1 \leq i \leq N_0\}$. For $x > M$,

$$F(x) = \sum_{x_n \leq x} p(x_n) \geq \sum_{n=1}^{N_0} p(x_n) > 1 - \epsilon$$

Let $m = \min\{x_i : 1 \leq i \leq N_0\}$. For $x < m$,

$$F(x) = \sum_{x_n \leq x} p(x_n) \leq \sum_{n=N_0+1}^{\infty} p(x_n) < \epsilon$$

As $\sum_{n=1}^{\infty} p(x_n) = 1$, $0 \leq F(x) \leq 1$ for all $x \in \mathbb{R}$. $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$.

(d) Let $A = \mathbb{Q}$. A is countable. Let $\{x_n\}_{n=1}^{\infty}$ be any enumeration of A. Let $p(x_n) = \frac{1}{e(n-1)!}$. $p(x_n) > 0$ for all $n \in \mathbb{N}$.

$$\sum_{n=1}^{\infty} p(x_n) = \frac{1}{e} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = 1$$

Hence $p(x_n) \leq 1$ for all $n \in \mathbb{N}$. The set of discontinuites of the F corresponding to this choice of A and p is $\{x \in A : p(x) \neq 0\} = \mathbb{Q}$. Let $q \in \mathbb{Q}$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. $\frac{1}{N} \in \mathbb{Q}$, therefore $q + \frac{1}{N} \in \mathbb{Q}$. Also, $\frac{1}{N} < \epsilon$. Hence q is a limit point of $\mathbb{Q}$. As every point of $\mathbb{Q}$ is a limit point of $\mathbb{Q}$, it contains no isolated points.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Suppose $c \in \mathbb{R}$ and $f(c) > 0$. Show that there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

(b) Consider $Z = \{x \in \mathbb{R} : f(x) = 0\}$. Show that Z contains all its limit points.

Solution by Prateek Karandikar:

(a) Let $\epsilon = \frac{f(c)}{2}$. By continuity of f, there exists a $\delta > 0$ such that for all $x \in (c-\delta, c+\delta)$, $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$. Therefore for all $x \in (c - \delta, c + \delta)$, $f(x) \in \left(\frac{f(c)}{2}, \frac{3f(c)}{2}\right)$, and so $f(x) > 0$.

(b) Let a be any limit point of Z. Choose a sequence a_n such that for all $n \in \mathbb{N}$, $a_n \in Z$ and $|a_n - a| < \frac{1}{2^n}$. Such a sequence exists as a is a limit point of Z. Then $\lim_{n\to\infty} a_n = a$. By continuity of f, $\lim_{n\to\infty} f(a_n) = f(a)$. For all $n \in \mathbb{N}$, $f(a_n) = 0$. Hence $f(a) = 0$ and $a \in Z$.