1. Assume $a_n > 0$ and that $\sum_{n=1}^{\infty} a_n < \infty$. Does it imply that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} < \infty$?

Solution by Prateek Karandikar:

Yes. For all $n \in \mathbb{N}$, define $b_n = \sqrt{a_n a_{n+1}}$ and $c_n = (a_n + a_{n+1})/2$. Note that

$$c_n - b_n = \frac{a_n + a_{n+1} - 2\sqrt{a_n a_{n+1}}}{2} = \frac{(\sqrt{a_n} - \sqrt{a_{n+1}})^2}{2} \geq 0$$

Therefore $0 \leq b_n \leq c_n$ (AM-GM inequality). Define the partial sum sequences $S_n = \sum_{k=1}^{n} a_n$ and $T_n = \sum_{k=1}^{n} c_n$.

$$\begin{aligned} T_n &= \sum_{k=1}^{n} \frac{a_n + a_{n+1}}{2} \\ &= \sum_{k=1}^{n} \frac{a_n}{2} + \sum_{k=2}^{n+1} \frac{a_n}{2} \\ &= \frac{a_1}{2} + \sum_{k=2}^{n} a_n + \frac{a_{n+1}}{2} \\ &= S_n - \frac{a_1}{2} + \frac{a_{n+1}}{2} \end{aligned}$$

As $\sum_{n=1}^{\infty} a_n$ is convergent, $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} S_n$ exists in $\mathbb{R}$. Hence

$$\begin{aligned} \lim_{n\to\infty} T_n &= \lim_{n\to\infty} S_n - \frac{a_1}{2} + \lim_{n\to\infty} \frac{a_{n+1}}{2} \\ &= \lim_{n\to\infty} S_n - \frac{a_1}{2} \end{aligned}$$

As $\lim_{n\to\infty} T_n$ exists in $\mathbb{R}$, the series $\sum_{n=1}^{\infty} c_n$ is convergent. $0 \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, hence by the comparion test $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} < \infty$.

□

3. Let a_n be a sequence of real numbers and a_{n_k} be a subsequence of the same. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Does it imply that $\sum_{k=1}^{\infty} a_{n_k}$ converges?

Solution by Prateek Karandikar:

No. Here is a counterexample. Let $a_n = (-1)^n/n$. We have seen in class that $\sum_{n=1}^{\infty} a_n$ converges. Let $n_k = 2k$. Then

$$a_{n_k} = a_{2k} = \frac{(-1)^{2k}}{2k} = \frac{1}{2k}$$

As $\sum_{k=1}^{\infty} 1/k$ does not converge, $\sum_{k=1}^{\infty} 1/2k$ also does not converge.

□

6. Let $p \in \mathbb{R}$. Decide whether $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{(n+1)^p}$ converges.

Solution by Prateek Karandikar:

Let $a_n = (-1)^n \frac{\sqrt{n}}{(n+1)^p}$. We will treat the two cases $p \leq \frac{1}{2}$ and $p > \frac{1}{2}$ separately.

First consider the case $p \leq \frac{1}{2}$. As $n + 1 \geq 1, (n+1)^p \leq (n+1)^{1/2}$. Also,

$$\begin{aligned} \frac{n}{n+1} - \frac{1}{2} &= \frac{n-1}{2n+2} \geq 0 \\ \frac{n}{n+1} &\geq \frac{1}{2} > 0 \\ \sqrt{\frac{n}{n+1}} &\geq \sqrt{\frac{1}{2}} \end{aligned}$$

We will now show that $\{a_n\}_{n=1}^{\infty}$ does not converge to 0.

$$\begin{aligned} |a_n| &= \frac{\sqrt{n}}{(n+1)^p} \\ &\geq \sqrt{\frac{n}{n+1}} \\ &\geq \sqrt{\frac{1}{2}} \\ &> 0 \end{aligned}$$

As $|a_n|$ is greater than or equal to a constant greater than 0 for all $n \in \mathbb{N}$, $\{a_n\}_{n=1}^{\infty}$ cannot converge to 0. Hence if $p \leq \frac{1}{2}$ the series $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{(n+1)^p}$ does not converge.

Now consider the case $p > \frac{1}{2}$. Define

$$b_n = (-1)^n \sqrt{\frac{n}{n+1}}, c_n = \frac{1}{(n+1)^{p-\frac{1}{2}}}$$

Therefore $a_n = b_n c_n$. As $p - \frac{1}{2} > 0$, $\lim_{n \to \infty} c_n = 0$. $c_n > 0$. Note that

$$\frac{c_{n+1}}{c_n} = \left(\frac{n+1}{n+2}\right)^{p-\frac{1}{2}} \leq 1$$

Hence $\{c_n\}_{n=1}^{\infty}$ is decreasing. We will now show that the partial sum sequence of $\{b_n\}_{n=1}^{\infty}$ is bounded. Define

$$d_n = \sqrt{\frac{n}{n+1}}$$

$b_n = (-1)^n d_n$ and $0 \leq d_n \leq 1$.

$$\begin{aligned}
\frac{n+1}{n+2} - \frac{n}{n+1} &= \frac{(n^2+2n+1)-(n^2+2n)}{(n+1)(n+2)} \\
&= \frac{1}{(n+1)(n+2)} \\
&> 0 \\
\frac{n+1}{n+2} &> \frac{n}{n+1} \\
\sqrt{\frac{n+1}{n+2}} &> \sqrt{\frac{n}{n+1}} \\
d_{n+1} &> d_n
\end{aligned}$$

Hence $\{d_n\}_{n=1}^{\infty}$ is increasing. Define the partial sum sequence of $\{b_n\}_{n=1}^{\infty}$, $S_n = \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (-1)^k d_k$. In what follows we will use the facts that $0 \leq d_n \leq 1$ and that $\{d_n\}_{n=1}^{\infty}$ is increasing.

$$\begin{aligned}
S_{2n} &= \sum_{k=1}^{n} (-d_{2k-1} + d_{2k}) \\
&\geq 0 \\
S_{2n} &= -d_1 + \sum_{k=1}^{n-1} (d_{2k} - d_{2k+1}) + d_{2n} \\
&\leq d_{2n} \\
&\leq 1 \\
|S_{2n-1} - S_{2n}| &= d_{2n} \\
&\leq 1
\end{aligned}$$

As every $k \in \mathbb{N}$ equals $2n$ or $2n-1$ for some $n \in \mathbb{N}$, we conclude that $|S_k| \leq 2$ for all $k \in \mathbb{N}$. So we have

1. The partial sum sequence of $\{b_n\}_{n=1}^{\infty}$ is bounded.

2. $c_{n+1} \leq c_n$ for all $n \in \mathbb{N}$.

3. $\lim_{n\to\infty} c_n = 0$.

Hence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n c_n$ is convergent for $p > \frac{1}{2}$.

$\square$