1(b).Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers. Let $t_n = \inf\{x_k : k \geq n\}$. Show that $t_n \to t$ for some $t \in \mathbb{R}$ and $t = \liminf_{n\to\infty} x_n$.

*Solution by Prateek Karandikar:* As $\{x_n\}_{n=1}^{\infty}$ is bounded, there exists $M > 0 \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Note that

$$\begin{aligned} t_n &= \inf\{x_k : k \geq n\} \\ &= \min(\inf\{x_k : k \geq n+1\}, x_n) \\ &= \min(t_{n+1}, x_n) \\ &\leq t_{n+1} \end{aligned}$$

Hence $\{t_n\}_{n=1}^{\infty}$ is an increasing sequence. $\{t_n\}_{n=1}^{\infty}$ is also bounded above by $M$. Therefore $\{t_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$. Let $t = \lim_{n\to\infty} t_n$.

We will now show that $t$ is a limit point of $\{x_n\}_{n=1}^{\infty}$. Set $k_0 = 0$. Suppose $k_0 < k_1 < \cdots < k_n \in \mathbb{N} \cup \{0\}$ have been chosen. As $t_{k_n+1} = \inf\{x_j : j \geq k_n + 1\}$, there exists $k_{n+1} \geq k_n + 1$ such that $0 \leq x_{k_{n+1}} - t_{k_n+1} < 2^{-n}$.

$\{t_{k_n+1}\}_{n=1}^{\infty}$ is a subsequence of $\{t_n\}_{n=1}^{\infty}$ and hence converges to $t$. Define a sequence $\{y_n\}_{n=1}^{\infty}$ by $y_n = x_{k_{n+1}} - t_{k_n+1}$. As shown above $0 \leq y_n < 2^{-n}\ \forall\ n \in \mathbb{N}$. So $\{y_n\}_{n=1}^{\infty}$ converges to 0.

$$\begin{aligned} x_{k_{n+1}} &= t_{k_n+1} + y_n \\ \lim_{n\to\infty} x_{k_{n+1}} &= \lim_{n\to\infty} t_{k_n+1} + \lim_{n\to\infty} y_n \\ \lim_{n\to\infty} x_{k_{n+1}} &= t + 0 \end{aligned}$$

Note that the second step above uses the fact that the sum of two convergent sequences always converges, and converges to the sum of the limits. $\{x_{k_{n+1}}\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ converging to $t$, hence $t$ is a limit point of $\{x_n\}_{n=1}^{\infty}$.

Let $\epsilon > 0 \in \mathbb{R}$. There exists a $N_0 \in \mathbb{N}$ such that $|t_n - t| = t - t_n < \epsilon$ for all $n \geq N_0$. Therefore $x_n \geq t_n > t - \epsilon$ for all $n \geq N_0$. So we conclude that $t = \liminf_{n\to\infty} x_n$. As $\epsilon > 0$ was arbitrary we have shown that $s < t$ then there is a $N_0$ such that $x_n > s$ for all $n \geq N_0$. $\square$

2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose the sequence is not bounded. Then show that there is either a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to \infty$ or a subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ such that $x_{m_k} \to -\infty$.

*Solution by Prateek Karandikar :* It is given that the sequence $\{x_n\}_{n=1}^{\infty}$ is not bounded above or not bounded below.

First consider the situation when $\{x_n\}_{n=1}^{\infty}$ is not bounded above. For every $M \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $x_n > M$. We will now construct a subsequence of $\{x_n\}_{n=1}^{\infty}$ whose limit is $\infty$. Set $n_1 = 1$. Suppose $n_1 < n_2 < \cdots < n_k \in \mathbb{N}$ have been chosen. Let $M = \max\{x_i : 1 \leq i \leq n_k\} + 1$. Note that we are taking the maximum of a finite non-empty set. There exists $p \in \mathbb{N}$ such that $x_p > M$. $p$ must be greater than $n_k$, for if $p \leq n_k$, then $x_p > M \geq x_p + 1$, which is a contradiction. Hence $p > n_k$. Set $n_{k+1} = p$.

Note that for all $k \in \mathbb{N}, x_{n_{k+1}} > x_{n_k} + 1$. Hence $x_{n_k} \geq x_1 + k - 1$. Let $A \in \mathbb{R}$. Choose $K_0 \in \mathbb{N}$ such that $K_0 > A - x_1 + 1$. For all $k \geq K_0$,

$$\begin{aligned} x_{n_k} &\geq x_1 + k - 1 \\ &\geq x_1 + K_0 - 1 \\ &> A \end{aligned}$$

Hence $\lim_{k\to\infty} x_{n_k} = \infty$.

Now consider the case when $\{x_n\}_{n=1}^{\infty}$ is not bounded below. For every $M \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $x_n < M$. We will now construct a subsequence of $\{x_n\}_{n=1}^{\infty}$ whose limit is $-\infty$. Set $m_1 = 1$. Suppose $m_1 < m_2 < \cdots < m_k \in \mathbb{N}$ have been chosen. Let $M = \min\{x_i : 1 \leq i \leq m_k\} - 1$. Note that we are taking the minimum of a finite non-empty set. There exists $p \in \mathbb{N}$ such that $x_p < M$. $p$ must be greater than $m_k$, for if $p \leq m_k$, then $x_p < M \leq x_p - 1$, which is a contradiction. Hence $p > m_k$. Set $m_{k+1} = p$.

Note that for all $k \in \mathbb{N}, x_{m_{k+1}} < x_{m_k} - 1$. Hence $x_{m_k} \leq x_1 - k + 1$. Let $A \in \mathbb{R}$. Choose $K_0 \in \mathbb{N}$ such that $K_0 > x_1 - A + 1$. For all $k \geq K_0$,

$$\begin{aligned} x_{m_k} &\leq x_1 - k + 1 \\ &\leq x_1 - K_0 + 1 \\ &< A \end{aligned}$$

Hence $\lim_{k\to\infty} x_{m_k} = -\infty$.

7. Suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Decide whether the series $\sum_{n=1}^{\infty} z_n$

converges in each of the following cases:

$$\begin{aligned} \text{(i)} \quad z_n &= \frac{\sqrt{n}}{2n^3-1} \\ \text{(ii)} \quad z_n &= \left(\frac{n}{2n+1}\right)^n \\ \text{(iii)} \quad z_n &= \frac{n^2-n+1}{n^3+1} \end{aligned}$$

*Solution by Prateek Karandikar:*

**(i)** $z_n = \dfrac{\sqrt{n}}{2n^3-1}$

For $n \in \mathbb{N}$,

$$1 - \frac{1}{2-\frac{1}{n^3}} = 1 - \frac{n^3}{2n^3-1} = \frac{2n^3-1-n^3}{2n^3-1} = \frac{n^3-1}{2n^3-1} \geq 0.$$

Hence $\frac{1}{2-\frac{1}{n^3}} \leq 1$. So

$$0 \leq z_n = \frac{\sqrt{n}}{2n^3-1} = \frac{1}{n^{2.5}} \frac{1}{2-\frac{1}{n^3}} \leq n^{-2.5}, \ \forall n \in \mathbb{N}.$$

As $\sum_{n=1}^{\infty} n^{-2.5}$ converges, by the comparison test, $\sum_{n=1}^{\infty} z_n$ also converges.

**(ii)** $z_n = \left(\dfrac{n}{2n+1}\right)^n$

For $n \in \mathbb{N}$,

$$\begin{aligned} |z_n|^{1/n} &= {z_n}^{1/n} \\ &= \frac{n}{2n+1} \\ &= \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2n+1} \end{aligned}$$

As $\lim_{n\to\infty} |z_n|^{1/n}$ exists and is equal to $\frac{1}{2}$ we have,

$$\limsup_{n\to\infty} |z_n|^{1/n} = \lim_{n\to\infty} |z_n|^{1/n} = \frac{1}{2} < 1$$

Therefore by the root test, $\sum_{n=1}^{\infty} z_n$ converges.

**(iii)** $z_n = \dfrac{n^2 - n + 1}{n^3 + 1}$

For $n \in \mathbb{N}$ such that $n \geq 2$,

$$\begin{aligned}
n &\geq 2 \\
n^3 &\geq 2n^2 \\
n^3 + 2n &\geq 2n^2 + 1 \\
n^3 - 2n^2 + 2n - 1 &\geq 0 \\
\frac{n^3 - 2n^2 + 2n - 1}{2n^3 + 2} &\geq 0 \\
\frac{n^3 - n^2 + n}{n^3 + 1} - \frac{1}{2} &\geq 0 \\
\frac{n^3 - n^2 + n}{n^3 + 1} &\geq \frac{1}{2}
\end{aligned}$$

Note that the last inequality holds for $n = 1$ also. For all $n \in \mathbb{N}$,

$$\begin{aligned}
z_n &= \frac{n^2 - n + 1}{n^3 + 1} \\
&= \frac{1}{n} \cdot \frac{n^3 - n^2 + n}{n^3 + 1} \\
&\geq \frac{1}{n} \cdot \frac{1}{2} \\
&\geq 0
\end{aligned}$$

As $\sum_{n=1}^{\infty} n^{-1}$ diverges, $\sum_{n=1}^{\infty} (2n)^{-1}$ also diverges, and hence by the comparison test $\sum_{n=1}^{\infty} z_n$ diverges. □