2(d). Decide (from definition) whether the following sequence $\{y_n\}_{n=1}^{\infty}$ is Cauchy or not:

$$y_n = \sum_{k=1}^{n} \frac{1}{k!}$$

Solution by Prateek Karandikar: We will show that this sequence is Cauchy. For $p \in \mathbb{N}, 1/p! \leq 1/2^{p-1}$. For $n, m \in \mathbb{N}$ and $n > m$,

$$\begin{aligned} y_n - y_m &= \sum_{k=m+1}^{n} \frac{1}{k!} \\ &\leq \sum_{k=m+1}^{n} \frac{1}{2^{k-1}} \\ &= \frac{\frac{1}{2^m}\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}{1 - \frac{1}{2}} \\ &= \frac{1}{2^{m-1}}\left(1 - \left(\frac{1}{2}\right)^{n-m}\right) \\ &\leq \frac{1}{2^{m-1}} \\ &< \frac{1}{2^m} \end{aligned}$$

Let $\epsilon > 0 \in \mathbb{R}$. Choose $N_0 \in \mathbb{N}$ such that $N_0 > 1/\epsilon$. As $2^{N_0} \geq N_0$, we have $2^{N_0} > 1/\epsilon$. For any $m, n > N_0$,

$$|y_n - y_m| < \frac{1}{2^{\min(m,n)}} < \frac{1}{2^{N_0}} < \epsilon$$

Hence the sequence is Cauchy. □

6(a). Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of real numbers and let $\{z_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $z_n = x_n + y_n$ for all $n \in \mathbb{N}$. Show that

$$\liminf_{n\to\infty} z_n \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$$

provided that the sum on the right hand side is not of the form $\infty - \infty$.

Solution: Let $X = \liminf_{n\to\infty} x_n, Y = \liminf_{n\to\infty} y_n, Z = \liminf_{n\to\infty} z_n$.

Case 1: Both X and Y are finite

For any $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that

$$x_n \geq X - \epsilon/2 \quad \forall n \geq N,$$

and

$$y_n \geq Y - \epsilon/2 \quad \forall n \geq N.$$

This implies that

$$z_n = x_n + y_n \geq X + Y - \epsilon \quad \forall n \geq N.$$

Suppose $Z < X + Y$. Let $\epsilon = (X + Y - Z)/2$. As Z is a limit point of $\{z_n\}_{n=1}^{\infty}$, there exists a subsequence n_p and $P \in \mathbb{N}, P > N$ such that $|z_{n_p} - Z| < \epsilon$ for all $p \geq P$. This implies,

$$z_{n_p} < X + Y - \epsilon, \quad \forall p \geq P.$$

This is a contradiction, and hence $Z \geq X + Y$.

Case 2: At least one of X and Y is infinite

Assume without loss of generality that X is infinite. It is given that $\{X, Y\} \neq \{\infty, -\infty\}$. If $X = -\infty$, then $X + Y = -\infty$. In this case, clearly $Z \geq X + Y$. Now consider the case when $X = \infty$. As $Y \neq -\infty$, the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded below by, say, M. For, if it did not have a lower bound, it would have a subsequence converging to $-\infty$, which contradicts the fact that $Y \neq -\infty$.

As $\limsup_{n\to\infty} x_n \geq X$, $\limsup_{n\to\infty} x_n = \infty = X$. Hence $\{x_n\}_{n=1}^{\infty}$ converges to ∞. For all $P \in \mathbb{R}$, there exists a $N_0 \in \mathbb{N}$ such that $x_n > P - M$ for all $n \geq n_0$. $x_n > P - M$ implies $x_n + M > P$, which implies $x_n + y_n > P$. Hence for all $P \in \mathbb{R}$, there exists a $N_0 \in \mathbb{N}$ such that $z_n > P$ for all $n \geq n_0$. This shows that $\{z_n\}_{n=1}^{\infty}$ converges to ∞ and so $Z = \infty$. Hence $Z \geq \infty = X + Y$.

$\square$