Question 5: If $a \in \mathbb{R}$ such that $0 \leq a < \epsilon$ for every $\epsilon > 0$, then show that $a = 0$.

Solution by Prateek Karandikar: We will prove this by contradiction. Suppose $a \neq 0$. Then, since it is given that $a \geq 0$, $a > 0$. By taking $\epsilon = a$, we conclude that $a < a$, which is a contradiction. Hence $a = 0$. □

Question 7: Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that

$$\sup(A + B) = \sup(A) + \sup(B)$$

and

$$\inf(A + B) = \inf(A) + \inf(B).$$

Solution by Prateek Karandikar: We will first prove that $\sup(A + B) = \sup(A) + \sup(B)$. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. As $\alpha \geq a \ \forall \ a \in A$ and $\beta \geq b \ \forall \ b \in B$, we conclude that $\alpha + \beta \geq a + b \ \forall \ a \in A, b \in B$. Hence $\alpha + \beta \geq x \ \forall \ x \in (A + B)$. So $\alpha + \beta$ is an upper bound of $A + B$.

Let $\gamma < \alpha + \beta$. Let $\epsilon = (\alpha + \beta - \gamma)/2$. Note that $\epsilon > 0$. As $\alpha - \epsilon$ is not an upper bound of A, $\exists \ a' \in A \ni a' > \alpha - \epsilon$. As $\beta - \epsilon$ is not an upper bound of B, $\exists \ b' \in B \ni b' > \beta - \epsilon$. Now, $a' + b' > (\alpha - \epsilon) + (\beta - \epsilon) = \gamma$. So we have found $x = a' + b' \in A + B$ such that $x > \gamma$.

Hence, $\sup(A + B) = \alpha + \beta = \sup(A) + \sup(B)$.

For any $X \subseteq \mathbb{R}$, define $-X := \{-x : x \in X\}$. So, for any $y \in \mathbb{R}$, $y \in X \iff (-y) \in (-X)$.

$$\begin{aligned}
(-A) + (-B) &= \{x + y : x \in (-A), y \in (-B)\} \\
&= \{(-x) + (-y) : x \in A, y \in B\} \\
&= \{-(x + y) : x \in A, y \in B\} \\
&= \{-z : z \in (A + B)\} \\
&= -(A + B)
\end{aligned}$$

We know that if $X \subseteq \mathbb{R}$ is non-empty and bounded below, $\inf(X) = -\sup(-X)$. Hence,

$$\begin{aligned}
\inf(A + B) &= -\sup(-(A + B)) \\
&= -\sup((-A) + (-B)) \\
&= -(\sup(-A) + \sup(-B)) \\
&= -\sup(-A) - \sup(-B) \\
&= \inf(A) + \inf(B)
\end{aligned}$$

□

Question 9: Show that the set $\mathbb{Z}$ of integers is countable.

Solution: Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n/2 & \text{if } n \in \mathbb{N} \text{ and is even} \\ -(n-1)/2 & \text{if } n \in \mathbb{N} \text{ and is odd,} \end{cases}$$

We will now show that f is a bijection. Suppose that for some $n_1, n_2 \in \mathbb{N}, f(n_1) = f(n_2)$. This implies

$$\frac{n_1}{2} = \frac{n_1}{2} \Rightarrow n_1 = n_2.$$

So, f is one-one. Let $m \in \mathbb{Z}$ and $m \leq 0$. Then $-2m + 1 \in \mathbb{N}$, is odd and $f(-2m + 1) = m$. Suppose $m \in \mathbb{Z}$ and $m > 0$. Then $2m \in \mathbb{N}$, is even and $f(2m) = m$. Therefore f is onto. As f is a bijection from $\mathbb{N}$ to $\mathbb{Z}$, $\mathbb{Z}$ is countable.