Question 5: Let A be a non-empty set of real numbers which is bounded below. Let $-A := \{x \in \mathbb{R} : -x \in A\}$. Show that $\inf(A) = -\sup(-A)$.

[*Solution by Prateek Karandikar*]: A is non-empty and bounded below, so it has a greatest lower bound. Let $\alpha = \inf(A)$.

As α is a lower bound of A, $\alpha \leq x \quad \forall\, x \in A$. Therefore $-\alpha \geq -x \;\forall\; x \in A$. As $x \in A \iff -x \in -A$, we conclude that $-\alpha \geq x \;\forall\; x \in -A$. This shows that $-\alpha$ is an upper bound of $-A$.

Let $\gamma < -\alpha$. To show that $-\alpha$ is the least upper bound of $-A$, we need to show that $\exists\; y \in -A \ni y > \gamma$. Let $\beta = -\gamma$. So $\beta > \alpha$. As α is the greatest lower bound of A, $\Rightarrow \exists\; x \in A \ni x < \beta$. So $\exists\; x \in A \ni x < -\gamma$. As $-x \in -A$ and $-x > \gamma$, we have found a $y \in -A$, namely $-x$, such that $y > \gamma$.

So we conclude that $-\alpha$ is the least upper bound of $-A$. So $\inf(A) = \alpha = -(-\alpha) = -\sup(-A)$.

$\square$

Question 7: If $z, w, z_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$ then show that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and

$$||z| - |w|| \leq |z - w|$$

[*Solution by Prateek Karandikar*]: We will prove the first inequality by induction on n. Let $\mathcal{P}(n)$ be the statement:

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|,$$

where $z_i \in \mathbb{C}$ for $i = 1, 2, \ldots n$.

Now,$\mathcal{P}(1)$ is true as $\mid z_1 \mid = \mid z_1 \mid$. We have seen in class that $\mathcal{P}(2)$ is true. Let $\mathcal{P}(k)$ be true for some $k \in \mathbb{N}$ such that $k \geq 2$. We will show that $\mathcal{P}(k+1)$ is true. Let $z_1, \ldots, z_k, z_{k+1} \in \mathbb{C}$.

$$\begin{aligned}
\left|\sum_{i=1}^{k+1} z_i\right| &= \left|\left(\sum_{i=1}^{k} z_i\right) + z_{k+1}\right| \\
&\leq \left|\sum_{i=1}^{k} z_i\right| + |z_{k+1}| \\
&\leq \left(\sum_{i=1}^{k} |z_i|\right) + |z_{k+1}| = \sum_{i=1}^{k+1} |z_i|,
\end{aligned}$$

where in the second step we have used $\mathcal{P}(2)$ and in the last step we have used the inductive hypothses. This proves $\mathcal{P}(k+1)$. Hence by the principle of mathematical induction, $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

Now we will prove the second inequality. For any $a, b \in \mathbb{R}$,

$$|a+ib| = \sqrt{a^2+b^2} = \sqrt{(-a)^2+(-b)^2} = |-(a+ib)|$$

Hence $|z_1| = |-z_1| \ \forall \ z \in \mathbb{C}$. In particular, $|z-w| = |w-z|$. Now,

$$|z| = |(z-w)+w| \leq |z-w| + |w|$$

$$|z| - |w| \leq |z-w|$$

Interchanging w and z,

$$|w| - |z| \leq |w-z| = |z-w|$$

Since $|w|$ and $|z|$ are reals, $||z|-|w||$ equals at least one of $|z|-|w|$ and $|w|-|z|$, both of which are less than or equal to $|z-w|$. Hence,

$$||z|-|w|| \leq |z-w|.$$

$\square$