Question 3 : Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$. Show that A is bounded above but does not have a least upper bound.

[*Solution by Prateek Karandikar*]: We will first show that if $x \in \mathbb{Q}$ such that $x > 0$ and $x^2 > 2$, then x is an upper bound of A. x is clearly greater than all non-positive elements of A. For all positive $a \in A$, $a^2 < 2 < x^2$, and so $a < x$. Hence $a \leq x \; \forall \; a \in A$. Therefore x is an upper bound of A. A is bounded above, as can be seen by setting $x = 5$.

If possible, let A have a least upper bound, say q. Since $1 \in A$, $q \geq 1$, and hence $q > 0$. If $q^2 < 2$, then $q \in A$, and for it to be an upper bound of A, it must be the largest element in A. We have seen in class that A has no largest element, hence $q^2 \geq 2$. Also, q^2 cannot equal 2, since there exists no $t \in \mathbb{Q}$ such that $t^2 = 2$. Hence $q^2 > 2$.

Now, let

$$r = \frac{q^2 + 2}{2q}$$

$$q - r = q - \frac{q^2 + 2}{2q} = \frac{q^2 - 2}{2q} > 0$$

The last inequality follows from the fact that $q^2 > 2$ and $q > 0$. So $r < q$.

$$r^2 - 2 = \frac{q^4 + 4q^2 + 4}{4q^2} - 2 = \frac{q^4 - 4q^2 + 4}{4q^2} = \left(\frac{q^2 - 2}{2q}\right)^2 > 0$$

As $r^2 > 2$ and $r > 0$, r is an upper bound of A. But $r < q$. This contradicts the assumption that q is the least upper bound of A, and so we conclude that A has no least upper bound. $\square$