Due: Tuesday, October 31st, 2006
Problems to be turned in: 5,6

1. Suppose A is a subset of $\mathbb{R}$ and c is a limit point of A. Show that there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in A$ and $x_n \to c$.

2. Let S be a subset of $\mathbb{R}$ that contains atleast two points and has the property that: if $x, y \in S$ then $[x, y] \subseteq S$. Show that S is an interval.

3. Let I be an interval. Let $f : I \to \mathbb{R}$ be continuous on I. Let $f(I) = \{f(x) : x \in I\}$. Suppose $k \in \mathbb{R}$ is such that $\inf f(I) \leq k \leq \sup f(I)$. Show that there exists a number $c \in I$ such that $f(c) = k$.

4. For each $x \in \mathbb{R}$, let $g^x : \mathbb{R} \to \mathbb{R}$ be given by $g^x(y) = 1_{(-\infty,x]}(y)$ (that is, $g^x(y) = 1$ if $y \leq x$ and 0 otherwise). Show that g^x is not continuous at x but is continuous otherwise.

5. Let A be a countable subset of $\mathbb{R}$. Consider $p : A \to [0, 1]$ such that $\sum_{n=1}^{\infty} p(x_n) = 1$ where $\{x_k : k \in \mathbb{N}\}$ is an enumeration of A. Define $F : \mathbb{R} \to [0, 1]$ by $F(x) = \sum_{x_n \leq x} p(x_n) \equiv \sum_{n=1}^{\infty} g^x(x_n)p(x_n)$.

   (a) Show that F is monotonically increasing.
   (b) Identify the discontinuity points of F and show that $F(x+) = F(x)$ for all $x \in \mathbb{R}$ (where $F(x+)$ is the notation for RHL at x).
   (c) Show that $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.
   (d) By choosing a suitable A and p construct an example of a monotonically increasing function whose points of discontinuity are not isolated.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

   (a) Suppose $c \in \mathbb{R}$ and $f(c) > 0$. Show that there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$
   (b) Consider $Z = \{x \in \mathbb{R} : f(x) = 0\}$. Show that Z contains all its limit points.

7. Find the continuity points of $f : \mathbb{R} \to \mathbb{R}$, when f is given by: (a) $f(x) = \lfloor x \rfloor$ (i.e. greatest integer less than or equal to x), (b) $f(x) = x\lfloor x \rfloor$, and (c) $f(x) = x - \lfloor x \rfloor$.

8. Show that $g(x) = \sqrt{x}$ is a uniformly continuous function on $[0, 1]$ but is not a Lipschitz function.