Due: Tuesday, October 24th, 2006
Problems to be turned in: 3,5,7

1. Let $a, b \in \mathbb{R}$. Show that $I = [a, b]$ is uncountable.
2. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose for $c \in \mathbb{R}$

$$\text{(LHL)} \quad L = \lim_{h \downarrow 0} f(c+h)$$

$$\text{(RHL)} \quad R = \lim_{h \downarrow 0} f(c-h)$$

(Notation: $h \downarrow 0$ means $h \in (0, \infty)$ and $h \to 0$)

   (a) Show that $\lim_{x \to c} f(x)$ exists if and only if LHL, RHL exists and are equal.
   (b) If f is continuous at c then show that $L = R = f(c)$.

3. Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Decide whether the RHL, LHL, and the limit exists for f at a given by: (a)$a = 0$, $f(x) = \frac{1}{x}, x \neq 0$, and $f(0) = 0$, (b) $a = -2$, $f(x) = \begin{cases} x+2 & x < -2 \\ -x-2 & x \geq -2 \end{cases}$, (c) $a = 0$, $f(x) = \frac{|x|}{x}, x \neq 0$, and $f(0) = 0$,
4. Let $f : (0, \infty) \to \mathbb{R}$ be given by:

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n}, (m, n) = 1, m, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   Show that f is continuous at every positive irrational number and discontinuos at every positive rational number.

5. (*Exponential Function:- e^x*) Consider the function $E : \mathbb{R} \to \mathbb{R}$ given by $E(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$.
   (a) Show that E is well defined and $E(x+y) = E(x)E(y)$, for all $x, y \in \mathbb{R}$.
   (b) Show that E is a continuous and monotonically increasing (strictly) function on $\mathbb{R}$.
   (c) Let $e = E(1)$. Show that $E(x) = e^x$ for all $x \in \mathbb{R}$.
   (d) $\lim_{x \to \infty} x^n e^{-x} = 0$ for all $n \in \mathbb{N}$.
6. (*Logarithm Function:-* $\ln(x)$) Let E be the function defined above. Let $L : (0, \infty) \to \mathbb{R}$ be such that:

$$L(E(y)) = y, \forall y \in \mathbb{R}.$$

   (a) Show that L is well defined and $L(uv) = L(u) + L(v)$, for all $u.v \in (0, \infty)$. ($L(x)$ is denoted by $\ln(x)$ for all $x > 0$)
   (b) Show that L is a continuous monotonically increasing (strictly) function.
   (c) Show that for any $\alpha \in \mathbb{R}$ $x \in [0, \infty)$, $x^\alpha = E(\alpha(L(x))) = e^{\alpha \ln(x)}$.
7. Let $\alpha \in (0, 1), A \subset \mathbb{R}$ A function $f : A \to \mathbb{R}$ is said to be Hölder-α if for all $x, y \in A$

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

   for some constant C.

   (a) Show that f is a continuous function on A.
   (b) Let $a, b \in \mathbb{R}$. If $f : [a, b] \to \mathbb{R}$ is such that $|f(x) - f(y)| \leq C|x - y|$, for all $x, y \in \mathbb{R}$ and for some constant C then show that f is Hölder-α for all $\alpha \in (0, 1)$ and consequently continous. Such functions are called Lipschitz functions.