Due: Wednesday, September 6th, 2006

Problems to be turned in: 1(b), 2,7

1. Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers. Let $s_n = \sup\{x_k : k \geq n\}$ and $t_n = \inf\{x_k : k \geq n\}$.
   (a) Show that $s_n \to s$ for some $s \in \mathbb{R}$ and further that s is a limit point of the sequence $\{x_n\}_{n=1}^{\infty}$.
   (b) Show that $t_n \to t$ for some $t \in \mathbb{R}$ and $t = \liminf_{n\to\infty} x_n$.
2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose the sequence is not bounded. Then show that there is either a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to \infty$ or a subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ such that $x_{m_k} \to -\infty$.
3. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers and E be its set of limit points. Show that $E = \{x\}$ for some $x \in \mathbb{R}$ if and only if $x_n \to x$.
4. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then show that

$$\liminf_{n\to\infty} \frac{z_{n+1}}{z_n} \leq \liminf_{n\to\infty} \sqrt[n]{z_n} \leq \limsup_{n\to\infty} \sqrt[n]{z_n} \leq \limsup_{n\to\infty} \frac{z_{n+1}}{z_n}$$

5. Suppose $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ are sequences of real numbers such that $x_n \leq z_n \leq y_n$ and the series $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ both converge. Does it always imply that $\sum_{n=1}^{\infty} z_n$ converges ?
6. Let $a, b, c \in \mathbb{R}$. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that

$$x_n = \frac{a}{n} + \frac{b}{n+1} + \frac{c}{n+2}, \quad \forall n \in \mathbb{N}$$

   Decide whether the series $\sum_{n=1}^{\infty} x_n$ converges or not.
7. Suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Decide whether the series $\sum_{n=1}^{\infty} z_n$ converges or not in each of the following cases:
   (i) $z_n = \frac{\sqrt{n}}{2n^3-1}$, (ii) $z_n = (\frac{n}{2n+1})^n$, (iii) $z_n = \frac{n^2-n+1}{n^3+1}$.
8. Let $C_1, C_2 \in \mathbb{R}\backslash\{0\}$ and $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences of real numbers. Suppose the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ both converge. Show that:
   (a) If $z_n = C_1 x_n + C_2 y_n$ then $\sum_{n=1}^{\infty} z_n$ also converges.
   (b) If $0 \leq z_n \leq C_1 x_n$ then $\sum_{n=1}^{\infty} z_n$ also converges.