Due: Tuesday, August 29th, 2006

Problems to be turned in: 2(d), 6(a)

1. Suppose $\{x_n\}_{n=1}^{\infty}$ is sequence of integers. Then show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is eventually a constant sequence (i.e. there is a $c \in \mathbb{R}, N \in \mathbb{N}$ such that $x_n = c$ for all $n \geq N$.)

2. Decide (from definition) whether the following sequences $\{y_n\}_{n=1}^{\infty}$ are Cauchy or not: (a) $y_n = (-1)^n$, (b) $y_n = \frac{(-1)^n}{n}$, (c) $y_n = \sum_{k=1}^{n} \frac{1}{k}$ and (d) $y_n = \sum_{k=1}^{n} \frac{1}{k!}$.

3. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that there is a constant $0 < C < 1$ such that $| x_{n+2} - x_{n+1} | \leq C | x_{n+1} - x_n |$, for all $n \in \mathbb{N}$. Then show that the sequence is Cauchy.

4. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $x_n \leq y_n$ for all $n \geq N$ for some fixed N, then show that
$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n.$$
$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n.$$

5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $S = \limsup_{n\to\infty} x_n$ and $I = \liminf_{n\to\infty} x_n$.
   (a) Prove or disprove: $| S | = \limsup_{n\to\infty} | x_n |$ and $| I | = \liminf_{n\to\infty} | x_n |$.
   (b) Suppose $S = I = x \in \mathbb{R}$ then show that $\lim_{n\to\infty} | x_n | = | x |$.

6. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of real numbers and let $\{z_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $z_n = x_n + y_n$ for all $n \in \mathbb{N}$. Show that
   (a) $\liminf_{n\to\infty} z_n \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$, provided the sum on the right hand side is not of the form $\infty - \infty$.
   (b) $\limsup_{n\to\infty} z_n \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$, provided the sum on the right hand side is not of the form $\infty - \infty$.

   Can you state and prove similar statements if $z_n = x_n y_n$ or $z_n = \frac{x_n}{y_n}$ with $y_n \neq 0 \forall n \in \mathbb{N}$. ?