Due: Tuesday, August 8th, 2006

Problems to be turned in: 5,7,9.

1. Let $f : \mathbb{R} \to \mathbb{R}$. be defined by $f(x) = x^2$. Then
   (a) Find $f(E)$, where $E = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$.
   (b) If $G := f(E)$, then find $f^{-1}(G)$ and $f(f^{-1}(G))$. (*Observe that* $f(f^{-1}(G)) = G$ *but* $f^{-1}(f(E)) \neq E$.)
2. Let $f : A \to B$. Let G be subset of B and $H \subset A$. Show that
$$f(f^{-1}(G)) \subset G, f^{-1}(f(H)) \supset H.$$
3. If $f : A \to B$ is injective and $g : B \to C$ is injective, then the composition $g \circ f : A \to C$ is injective.
4. If $a, b \in \mathbb{R}$ and $a < b$, then show that $a < \frac{a+b}{2} < b$.
5. If $a \in \mathbb{R}$ such that $0 \leq a < \epsilon$ for every $\epsilon > 0$, then show that $a = 0$.
6. Let $S \subset \mathbb{R}$, be bounded above. Show that $u = \sup(S)$ if and only if for every $\epsilon > 0$ there is a $s \equiv s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.
7. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup(A) + \sup(B)$ and $\inf(A + B) = \inf(A) + \inf(B)$.
8. Suppose $I_n = (0, \frac{1}{n})$ then show that $\cap_{n=1}^{\infty} I_n = \emptyset$.
9. Show that the set $\mathbb{Z}$ of integers is countable.
10. Let $y \in \mathbb{R}, x \in [0, 1]$. Show that there is a sequence $\{b_n : n \in \mathbb{N}\}$, such that $0 \leq b_n \leq 9$ and
$$\frac{b_1}{10} + \frac{b_2}{10^2} + \ldots + \frac{b_n}{10^n} \leq x \leq \frac{b_1}{10} + \frac{b_2}{10^2} + \ldots + \frac{b_n}{10^n} + \frac{1}{10^{n+1}}, \tag{1}$$
for all $n \in \mathbb{N}$. Conversely given a sequence $\{b_n : n \in \mathbb{N}\}$ such that $0 \leq b_n \leq 9$ there is a unique $x \in [0, 1]$ satisfying (1) for all $n \in \mathbb{N}$.
We shall then write $x := .b_1 b_2 b_3 \ldots$ *and call this as a decimal expansion for* x. *For any* $y \in \mathbb{R}$ *we shall refer to* $y := N.b_1 b_2 b_3 \ldots$ *as a decimal representation for* y, *where* $y = N + x$ *and* $N \in \mathbb{N} \in [0, 1]$. Is the above decimal representation unique for every x ?
11. Using the previous problem can you provide another definition of rational numbers $\mathbb{Q} \subset \mathbb{R}$ and a proof for the fact that $\mathbb{R}$ is uncountable ?