Not Due

1. Let $-\infty \leq a < b \leq \infty$ and $f : (a, b) \to \mathbb{R}$ be a differentiable function.

   (a) If $c \in (a, b)$ is a local maximum of f (i.e. there is a $\delta > 0$ such that $f(x) \leq f(c)$ whenever $|x - c| < \delta$ and $x \in (a, b)$.) then $f'(c) = 0$.

   (b) f is increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

   (c) Suppose $f'(c) = 0$ for all $c \in (a, b)$. From first principles, show that f is a constant function.

   (d) Can you construct a $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) \neq 0$ but f is not monotonic in any neighbourhood of 0 ?

2. Let $A \in \mathbb{R}$ and $-\infty \leq a < b \leq \infty$. Let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \text{ and } \lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

   (a) Deduce first that $g(x) \neq g(y)$ for all $a < x < y < b$.

   (b) Let $\epsilon > 0$ be given. Use the generalised mean value theorem to conclude that there is a $\delta > 0$ such that

$$L - \epsilon < \frac{f(x) - f(y)}{g(x) - g(y)} < L + \epsilon,$$

   for all $a < x < y < a + \delta < b$.

   (c) Show that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = A.$$

   (d) Show that (c) holds if $A \in \{\infty, -\infty\}$.

   (e) Suppose $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ is replaced by $\lim_{x \to a} g(x) = \infty$ show that (c) holds as well.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Show that f is differentiable at c if and only if there is a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(c) + \phi(x)(x - c)$$

4. Let I be an interval and $f : I \to \mathbb{R}$ be differentiable on I. Then f is increasing (decreasing) if and only if $f'(x) \geq 0 (f'(x) \leq 0)$ for all $x \in I$.

5. (*Darboux's Theorem*) If f is differentiable on $I = [a, b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point $c \in (a, b)$ such that $f'(c) = k$. Can you say something about the discontinuities of f' ?. (*Hint: consider* $g(x) = xk - f(x)$)

6. (*Bartle and Sherbert page 192*) Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be differentiable on I. Suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2 + 2x + 3$ then find its Taylor polynomial of degree n for all $n \in \mathbb{N}$ at $x_0 = 0$ and $x_0 = 1$.

8. Consider the convergent series $\sum_{n=1}^{\infty} c_n$. Show that $f(x) = \sum_{n=1}^{\infty} c_n x^n$ is well defined on $(-1, 1]$ and decide whether it is continuous on $(-1, 1]$.

9. Let $\{x_n\}_{n=1}^{\infty}$and $\{y_n\}_{n=1}^{\infty}$be a sequence of real numbers. Suppose $\limsup_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Show that $\limsup_{n\to\infty} x_n y_n = xy$

10. Suppose $f : (-R, R) \to \mathbb{R}$ and $f(x) = \sum_{n=1}^{\infty} c_n x^n$with $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \frac{1}{R}$. Then can you find the its Taylor series at $a \in \mathbb{R}$ (i.e. it exists and $f^n(a)$ interms of c_n) and also its radius of convergence. (*You may not be able to prove the result rigorously, given the theorems done in class.*)

11. Find the Taylor Polyonmial of degree n of the following functions $f : \mathbb{R} \to \mathbb{R}$ at $x_0 = 0$: (a) $f(x) = \ln(x)$ and (b) $f(x) = \cos(x)$.

12. Using Taylor's Theorem, find good rational bounds for $\sqrt{3}$.