Due: Tuesday, 14th November, 2006

1. Let $I = (a, b), a, b \in \mathbb{R}$ be an open interval. let $f : I \to \mathbb{R}$ be a continuous monotonically (strictly) increasing function. Show that $f(I)$ is also an open interval.

2. (*Rudin Exercise 6, chapter 1*). Fix $b > 1$.

   (a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = \frac{m}{n} = \frac{p}{q}$ prove that $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$. (Thereby we can define $b^r = (b^m)^{\frac{1}{n}}$.)

   (b) Prove that $b^{r+s} = b^r b^s$ if $r, s \in \mathbb{Q}$.

   (c) If $x \in \mathbb{R}$, define $B(x) = \{b^p : p \in \mathbb{Q}, p \le x\}$ Prove that $b^r = \sup B(r)$ if $r \in \mathbb{Q}$. (Thereby it makes sense to define $b^x = \sup B(x)$ for $x \in \mathbb{R}$)

   (d) Prove that $b^{x+y} = b^x b^y$ for all $x, y \in \mathbb{R}$.

   (e) Let $g : \mathbb{R} \to \mathbb{R}$ and $g(x) = b^x$. Show that g is continuous.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

   Show that f is continuous at 0 and LHD (left hand derivative) and RHD(Right hand derivative) of f do not exist at 0.

4. Let $E : \mathbb{R} \to \mathbb{R}$ be such that $E(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$. Show that E is differentiable and $E'(x) = E(x)$

5. Let $f, g : E \to \mathbb{R}$ Suppose x is a limit point of E. Suppose

$$\lim_{y \to x} f(y) = A, \lim_{y \to x} g(y) = B,$$

   with $A, B \in \mathbb{R} \cup \{\infty, -\infty\}$. Then

   (a) $\lim_{t \to x} f(t) = C$ implies $C = A$.

   (b) $\lim_{t \to x} f + g(t) = A + B$

   (c) $\lim_{t \to x} fg(t) = AB$

   (d) $\lim_{t \to x} \frac{f}{g}(t) = \frac{A}{B}$,

   provided all the operations are well-defined. (i.e no division by 0 or $\infty - \infty$ or $0 \cdot \infty$ or $\frac{\infty}{\infty}$)

6. (*Rudin Page 114 1.)* Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that

$$|f(x) - f(y)| \le (x - y)^2.$$

   Show that f is a constant function.

7. (*Bartle and Sherbert: page 167)* Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^{\frac{1}{3}}$. Show that f is not differentiable at $x = 0$.

8. (*Bartle and Sherbert: page 167)* Decide whether $h : \mathbb{R} \to \mathbb{R}$ and $h(x) = x|x|$ is differentiable or not.