Problems to be turned in: 7,10
Due: 22nd, October 2008

1. Define what is meant to say a space $\mathcal{H}$ is a separable Hilbert space. Show that in a separable Hilbert space any collection of pairwise orthogonal vectors is at most countable.

2. Let $\mathcal{P} = \{f \in C(S^1) |$ there exists $\{a_k\}_{k=-N}^{k=N}$ such that $f(x) = \sum_{k=-N}^{N} a_k e^{ikx}\}$. Show that $\mathcal{P}$ is dense in $L^2[-\pi, \pi]$.

3. For $n \geq 1$, let $E_n \in C(S^1)$ be defined as $E_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$ for $x \in [-\pi, \pi]$. If $f \in L^2[-\pi, \pi]$ and $\langle f, E_n \rangle = 0$ for all $n \geq 1$ then $f = 0$ a.e.

4. Provide an example of $T : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$ such that T is length preserving but T is not onto.

5. Let $(\Omega, \mathcal{B}, \mu)$ be a finite complex measure space. Define the total variation measure $|\mu| :\to [0, \infty]$ by
$$|\mu|(E) = \sup\{\sum_{n=1}^{\infty} |\mu(E_n)| : E = \coprod_{n=1}^{\infty} E_n, E_n \in \mathcal{B}, \forall n\}.$$
Show that $|\mu|$ is a finite positive measure on $(\Omega, \mathcal{B})$.

6. Suppose $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is monotonically increasing. Assuming that the derivative f' exists a.e. show that it is measurable.

7. Suppose μ is a complex Borel measure on $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{C}$ given by
$$f(x) = \mu((-\infty, x)) \quad (x \in \mathbb{R}).$$
For $x \in \mathbb{R}$ and $A \in \mathbb{C}$, then each of the following are equivalent:
(a)f is differentiable at x and $f'(x) = A$
(b) For every $\epsilon > 0$ there exists a $\delta > 0$ such that
$$\left| \frac{\mu(I)}{\lambda(I)} - A \right| < \epsilon,$$
where I is any open interval containing x and whose length is less than δ.

8. Let $a, b \in \mathbb{R}$ and $f, g : [a, b] \to \mathbb{R}$ be two absolutely continuous functions. Show that fg is also an absolutely continuous function. Can you now state and prove an integration by parts formula ?

9. Suppose $E \subset [a, b]$ and $\lambda(E) = 0$. Can you construct absolutely continous monotonic increasing function $f : [a, b] \to \mathbb{R}$ so that $f'(x) = \infty$ for all $x \in E$.

10. Let $a, b \in \mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be such that $\int f(x) 1_{[a,c]} dx \equiv \int_a^c f(x)dx = 0$ for all $c \in (a, b)$. Show that $f = 0$ a.e.