Due: October, 1st 2008
Problems to be turned in : 2,4

1. Let $\{A_1, \cdots, A_n\}$ be subsets of any set Ω. For $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{0,1\}^n$, define

$$A(\epsilon) = \cap_{i=1}^n A_i^{(\epsilon_i)}, \text{where we write } B^{(i)} = \begin{cases} B & if\ i = 1 \\ B^c & if\ i = 0 \end{cases}$$

   Show that

   (a) $\Omega = \coprod_{\epsilon \in \{0,1\}^n} A(\epsilon)$;

   (b) $A_i = \cup\{A(\epsilon) : \epsilon(i) = 1\}$;

   (c) $\{\epsilon : A(\epsilon) \neq \emptyset\}$ is the collection of 'atoms' (= minimal non-empty elements) of $\mathcal{A}(\{A_1, \cdots, A_n\})$; and

   (d) If $|\{\epsilon : A(\epsilon) \neq \emptyset\}| = k$, then $\mathcal{A}(\{A_1, \cdots, A_n\})$ has 2^k elements.

2. Let $\{f, f_n : n \geq 1\}$ be $\bar{\mathbb{R}}$-valued measurable functions on $(\Omega, \mathcal{A}, \mu)$. Suppose $f_n \uparrow f$ and that there exists $(\mathcal{A}, \mathcal{B}_{\bar{\mathbb{R}}})$- measurable function such that $\int h_- d\mu < \infty$ and $f_n \geq h$ for all $n \geq 1$. Then show that $\int f_n d\mu \uparrow \int f d\mu$.

3. Let $\Omega = \Omega_1 \times \Omega_2$; call a set of the form $A_1 \times A_2 \subseteq \Omega, A_i \subseteq \Omega_i$, a rectangle, and call such a rectangular measurable if $A_i \epsilon \mathcal{B}_i, i = 1, 2$.

   (a) Show that $\mathcal{A}(\mathcal{R})$, the algebra generated by the measurable rectangles, is precisely the collection of sets of the form $\coprod_{i=1}^n R_i, R_i \ \epsilon\ \mathcal{R}$.

   (b) Supppose for all $E \subset \Omega_1 \times \Omega_2, x \in \Omega_1$. Let $E^x = \{y \in \Omega_2 : (x,y) \in E\}$. Then show that for any sequence of disjoint sets $\{E_n\}_{n=1}^\infty$ in Ω, $(\cup_{n=1}^\infty E_n)^x = \cup_{n=1}^\infty E_n^x$

4. Let $\Omega_1 = \Omega_2 = \mathbb{N}$, $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{P}(\mathbb{N})$. Define the measures μ_i on $(\Omega_i, \mathcal{B}_i)$ by $\mu_i(\{k\}) = 2^{-k}$. Define $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ by

$$f(m,n) = \begin{cases} -n2^{2n} & \text{if } m = n \\ n2^{2n} & \text{if } m = n-1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that f is $\mathcal{B}_1 \otimes \mathcal{B}_2$ measurable.
   (b) Observe that

$$\int_{\Omega_2} \int_{\Omega_1} f(m,n) d\mu_1(m) d\mu_2(n) \neq \int_{\Omega_1} \int_{\Omega_2} f(m,n) d\mu_2(n) d\mu_1(m),$$

   thereby emphasising the importance of integrability in the hypotheses of Fubini's theorem.

5. Suppose $(\Omega, \mathcal{B}, \mu)$ is a σ -finite measure space and $f : \Omega \to \mathbb{R}$ is a non-negative $((\mathcal{B}, \mathcal{B}_{\mathbb{R}})-)$ measurable function. Define $\mathcal{G} = \{(w,t)\ \epsilon\ \Omega \times \mathbb{R} : 0 \leq t \leq f(w)\}$. Show that $\mathcal{G}\ \epsilon\ \mathcal{B} \otimes \mathcal{B}_{\mathbb{R}}$, and that $(\mu \times m)(\mathcal{G}) = \int f d\mu$, where m denotes Lebesgue measure on $\mathbb{R}$.