Problems to be turned in: 1, 4, 6, 12, 13, 15
Due: 18th August, 2008

1. If $\Omega \in \mathcal{A}$ and if $A, B \in \mathcal{A}$ implies that $A \cap B^c \in \mathcal{A}$, then show that $\mathcal{A}$ is an algebra.

2. If $\mathcal{A}$ is an algebra of sets in Ω, and if $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, then there exist sequences $\{S_n\}_{n=1}^{\infty}, \{D_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$ such that:

   (i) $S_1 \subset S_2 \subset S_3 \subset$ ....;
   (ii) $D_n \cap D_m = \emptyset$ if $n \neq m$; and
   (iii) $\cup_{k=1}^{n} E_k = \cup_{k=1}^{n} S_k = \cup_{k=1}^{n} D_k$, for $n = 1, 2, ...$

   Further, conditions (i) - (iii) uniquely determine the sets S_n and D_n, for all n.

3. Let $\mathcal{B}$ be a collection of subsets of a set Ω.

   (a) If $\mathcal{B}$ is a σ-algebra, then $\mathcal{B}$ is an algebra as well as a monotone class.

   (b) If $\mathcal{B}$ is an algebra as well as a monotone class, then $\mathcal{B}$ is a σ-algebra.

   (c) Show that if $\mathcal{S}$ is any class of subsets of Ω, then there exists a smallest algebra $\mathcal{A}(\mathcal{S})$ (resp. monotone class $\mathcal{M}(\mathcal{S})$ containing $\mathcal{S}$.

4. Let $\mathcal{A}$ be an algebra of subsets of a set Ω . Let $\Omega_0 \subseteq \Omega$. Define $\mathcal{A} \cap \Omega_0 = \{A \cap \Omega_o : A \in \mathcal{A}\}$. Show that $\mathcal{A} \cap \Omega_0$ is an algebra of subsets of Ω_0, and that $\mathcal{A} \cap \Omega_0$ is a σ-algebra if $\mathcal{A}$ is. (In the case when $\Omega_0 \in \mathcal{A}$, it is more natural and customary to write $\mathcal{A}|_{\Omega_0}$ instead of $\mathcal{A} \cap \Omega_0$.) What if $\mathcal{A}$ is a monotone class?

5. Let Ω be a non-empty set.

   (a) How many distinct algebras of subsets of Ω exist, if Ω is a three element set?

   (b) If $\mathcal{A}$ is a finite algebra of subsets of (a possibly infinite set) Ω, what can you say about the number of distinct sets in $\mathcal{A}$?

6. Let $\Omega = \mathbb{R}$, let $\mathcal{S}$ denote the collection of intervals of the form (a,b], where $-\infty \leq a < b \leq \infty$ (where of course $(a, \infty]$ is to be interpreted as (a, ∞)). Show that a typical non-empty element of $\mathcal{A}(\mathcal{S})$, the algebra generated by $\mathcal{S}$, is of the form $\coprod_{k=1}^{n} I_k$, where $n = 1, 2, \ldots$. and $I_k \in \mathcal{S}$ for $1 \leq k \leq n$.

7. Assume $\mathcal{B}$ is a σ-algebra on Ω. Let $B \subset \Omega$. Let $C = \mathcal{B} \cup \{B\}$. Show that $\sigma(C) = \{(B \cap U) \cup (B^c \cap V) : U, V \in \mathcal{B}\}$

8. Assume $\mathcal{B}$ is a σ-algebra on Ω. If $\{A_1, A_2, \ldots A_n\}$ is a partition of Ω, then describe $\sigma(\mathcal{B} \cup \{A_1, A_2, \ldots, A_n\})$.

9. Let $\mathcal{B}_1, \mathcal{B}_2$ be two σ-algebras on Ω. Show that $\sigma(\mathcal{B}_1 \cup \mathcal{B}_2) = \sigma(\mathcal{C})$ where $\mathcal{C} = \{A_1 \cup A_2 : A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2\}$.

10. Let $n \in \mathbb{N}$. The σ-algebra $\mathcal{B}^n$ on $\mathbb{R}^n$ generated by open subsets of $\mathbb{R}^n$, is called the Borel σ-algebra of $\mathbb{R}^n$, and members of $\mathcal{B}^n$ are called Borel sets .

    (a) For $B \subset \mathbb{R}$ and $x \in \mathbb{R}$, let $x + B = \{x + y : y \in B\}$. Show that for every element $B \in \mathcal{B}$, $x + B$ is also a Borel set (i.e. an element of the Borel σ-algebra).

    (b) Let $T : \mathbb{R} \to \mathbb{R}$ be defined as $T(x) = x + 1$. Show that $\tilde{\mathcal{B}} = \{B \in \mathcal{B} : TB = B\}$ is a σ-algebra on $\mathbb{R}$.

(c) More generally, if (X,d) is a meric space, and if $\mathcal{S}_1 = \{B(x,r) : x \in X, r > 0\}$ is the set of open balls in X, and if $\mathcal{S}_2$ is the set of open sets in X, verify that $\sigma(\mathcal{S}_1) = \sigma(\mathcal{S}_2)$ provided X is a separable metric space. In that case, this common σ-algebra is called the **Borel σ-algebra** of X, it is denoted by $\mathcal{B}_X$, and its members are called Borel sets.

(d) If $f : X \to X$ is a homeomorphism, then show that $E \in \mathcal{B}_X \Rightarrow f(E) \in \mathcal{B}_X$.

11. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Define $\mathcal{S}_1 = \{(a,b) : -\infty < a < b < \infty\}$, $\mathcal{S}_2 = \{[a,b] : -\infty < a < b < \infty\}$, $\mathcal{S}_3 = \{[a,b) : -\infty < a < b < \infty\}$, $\mathcal{U}$ = collection of all open sets in $\mathbb{R}$, $\mathcal{C}$= collection of all closed sets in $\mathbb{R}$, and $\mathcal{K}$ = collection of all compact sets in $\mathbb{R}$. Show that $\mathcal{B} = \sigma(\mathcal{S})$, where $\mathcal{S}$ is any one of the collections $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{U}, \mathcal{C}$ *or* $\mathcal{K}$.

12. Let $\Omega = (0,1]$ and $\mathcal{B}$ be the σ-algebra generated by open sets in Ω, i.e. the Borel σ-algebra on Ω. Show that $\tilde{\mathcal{B}} = \{B \subset \Omega : B \in \mathcal{B}$ and is either disjoint from $(\frac{1}{2}, 1]$ or contains $(\frac{1}{2}, 1]\}$ is a σ-algebra on Ω. Moreover $\tilde{\mathcal{B}} = \sigma($collection of all intervals contained in $(0, \frac{1}{2}))$.

13. Let $\mathcal{B}^n$ be the Borel σ-algebra in $\mathbb{R}^n$ for $n \in \mathbb{N}$.
(a) Show that $\mathcal{B}^1 = \sigma((a,b] : -\infty \le a \le b \le \infty)$
(b) Show that $\mathcal{B}^2 = \sigma((a,b] \times (c,d] : -\infty \le a \le b \le \infty, -\infty \le c \le d \le \infty)$
(c) Show that $\mathcal{B}_1 = \{B \times \mathbb{R} : B \in \mathcal{B}^1\}$ and $\mathcal{B}_2 = \{\mathbb{R} \times B : B \in \mathcal{B}^1\}$ are both σ-algebras on $\mathbb{R}^2$ contained in $\mathcal{B}^2$ and that $\mathcal{B}^2 = \sigma(\mathcal{B}_1 \cup \mathcal{B}_2)$.
(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $(x,y) \to (y,x)$. Show that $\tilde{\mathcal{B}} = \{B \in \mathcal{B}^2 : TB = B\}$ is a σ-algebra and find a generating set.

14. Suppose μ is a finitely additive set function defined on $\mathcal{A}$.

(i) Then, $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)\ \forall A, B \in \mathcal{A}$.

In particular, if $\mu(\Omega) < \infty$ (so that there is no problem with subtraction), then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

(ii) Can you write down a general formula for $\mu(A \cup B \cup C), A, B, C \in \mathcal{A}$, (or, more generally, for $\mu(\cup_{i=1}^n A_i)$, when $\{A_i\}_{i=1}^n \subseteq \mathcal{A}$) - under the assumption that $\mu(\Omega) < \infty$?

15. Let Ω be a countable set and $\mathcal{A} = 2^\Omega$ be the collection of all subsets of Ω. Then,

(i) $\mathcal{A}$ is a σ-algebra;

(ii) if $\mu : \mathcal{A} \to [0,\infty]$ is defined by $\mu(E)$ = 'number of elements in E', then μ is a measure, and is called **the counting measure on** Ω (since μ counts the number of elements in a set).

(iii) If $\Omega = \{w_1, w_2, \ldots\}$ is an enumeration of Ω, and if Ω is infinite, let $A_n = \{w_n, w_{n+1}, ...\}$; notice that $A_n \downarrow \emptyset$ but

$$\mu(\emptyset) = 0 \ne \infty = \lim_{n\to\infty} \mu(A_n).$$

(iv) If μ is a general possibly infinite measure defined on an algebra $\mathcal{A}$ of subsets of any set Ω, and if $A, A_n \in \mathcal{A}, A_1 \supseteq A_2 \supseteq, \ldots, A = \cap_{n=1}^\infty A_n$, then show that $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ provided there exists some k so that $\mu(A_k) < \infty$.