Due: August 11th, 2008

1. **Cardinality :** For any two non-empty sets X, Y We say "cardinality of X", written card(X), is less than or equal to card(Y) if there is an injection $f : X \to Y$.
   (a) Show for any two non-empty sets $\text{card}(X) \leq \text{card}(Y)$ or $\text{card}(Y) \leq \text{card}(X)$.
   (b) Show for any two non-empty sets $\text{card}(X) \leq \text{card}(Y)$ iff $\text{card}(Y) \geq \text{card}(X)$.
   (c) Show for any two non-empty sets $\text{card}(X) = \text{card}(Y)$ iff there is a bijection $\phi : X \to Y$.
   (d) A set X is countable if $\text{card}(X) \leq \text{card}(\mathbb{N})$.. $\text{card}(X) = n$ if and only if $\text{card}(X) = \text{card}(\{1, 2, \ldots, n\})$, such sets are called countably finite.
   (e) Let $c = \text{card}(\mathbb{R})$. Show that $\text{card}(P(\mathbb{N})) = c$.
   (f) Suppose X is a set such that $\text{card}(X) > c$ then X is uncountable.

2. **Real Numbers:** Let $\mathbb{R}$ be the set of real numbers and $\bar{R} = \{\infty\} \cup \mathbb{R} \cup \{-\infty\}$ be the extended real numbers.
   (a) A set U in $\mathbb{R}$ is said to be open if for every $x \in U$ there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Show that every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.
   (b) Define what is meant by a sequence in $\mathbb{R}$. Suppose $\{x_n\}$ is a sequence in $\mathbb{R}$ then define the concept of limit, limit superior and limit inferior. For any two sequences $\{x_n\}$, $\{y_n\}$ show that
   $$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$
   (c) Let X be an arbitrary set and $f : X \to [0, \infty]$. We define
   $$\sum_{x\in X} f(x) = \sup\{\sum_{x\in F} f(x) : F \subset X, F \text{ finite}\}.$$
   Let $A = \{x : f(x) > 0\}$. If A is uncountable, then show that $\sum_{x\in X} f(x) = \infty$. If A is countably infinite, then show that $\sum_{x\in X} f(x) = \sum_{n=1}^{\infty} f(g(n))$ where $g : \mathbb{N} \to A$ is any bijection and the sum on the right is an ordinary infinite series.

3. **Metric Spaces:** Let X, ρ be a metric space, $E \subset X$, and $x \in X$. Then show:
   (a) $x \in \bar{E}$
   $$\Leftrightarrow$$
   $B(r, x) \cap E \neq \emptyset$ for all $r > 0$
   $$\Leftrightarrow$$
   there is a sequence $\{x_n\}$ in E which converges to x.
   (b) Define what is meant by "E is complete and totally bounded". E is complete and totally bounded
   $$\Leftrightarrow$$
   every sequence in E has a subsequence which converges to a point of E
   $$\Leftrightarrow$$
   if $\{V_\alpha\}_{\alpha\in A}$ is a cover of E by open sets, there is a finite set $F \subset A$ such that $\{V_\alpha\}_{\alpha\in F}$ covers E.
   (c) Define what is meant by "X is a complete metric space". Suppose X is a complete metric space. Show that: E is a closed set $\Leftrightarrow$ E is a complete subset of X.
   (d) Suppose $X = \mathbb{R}^n$ equipped with the usual metric. Show that every closed and bounded subset of X is compact.