Triterm I: October 18th, 2000

Syllabus: 2.1-2.8,3.1-3.9,4.1,4.2

1. **Exam Instructions:**
   (a) The Exam will be closed book, no notes.
   (b) A simple calculator will be allowed...(i.e. one without programmable function of any kind and any other fancy stuff).
   (c) Please show your work in the exam, write in complete sentences, and Indicate your answers clearly.
   (d) The exam will have 5 questions and shall be worth 50 points.
2. **Preparing for the Triterm:**
   (a) You should know *at the very least* the definition/meaning/ of any of the following terms:
      i. Sample Space.
      ii. Events.
      iii. Mutually exclusive events.
      iv. Probability.
      v. Independent Events.
      vi. Conditional probability.
      vii. Rules of probability (Example: Q6 in Worksheet 1)
      viii. Discrete Random variable.
      ix. Probability Mass function, Distribution Function.
      x. Bernoulli/Binomial/Poisson/Geometric/Negative Binomial/Hypergeometric/ distributions. Their expectation and variance.
      xi. Moment generating function.
      xii. Continuous Random Variable.
      xiii. Probability Density function, Distribution Function.
   (b) You will need to understand Theorem 2.8 in the case k=2 only. You donot need to understand Theorem 3.4
   (c) You may ignore(for this triterm): the discussion about runs on page 28/29
   (d) You will need to understand the relationships between the Discrete Random variables (via there interpretations through experiments).
   (e) Do all the examples, worksheet problems, practice problems and homework problems. Try as many problems as you can from the text.

[1] **Office hours:** Monday: 2:00-3:00 pm, Tuesday:10:30-11:30, 3:30-4:30 pm, Friday: 1:30-2:30pm

Practice midterm

1. Suppose that 10% of all children are left–handed.

   (a) In a class of 20 children, what is the probability that at least two are left–handed?

   (b) Suppose a certain school has 10 classes of 20 children. If you check the classes one by one, what is the probability that the first left–handed child will be found in the fourth class?

2. The number of raisins in a small cookie from a certain bakery has a Poisson distribution with $\lambda = 3$. Suppose we take two of these cookies.

   (a) What is the probability that each will contain at least one raisin?

   (b) What is the probability that the two cookies will have a total of at least two raisins?

3. A random variable has the density function

$$f(y) = \begin{cases} \frac{1}{y^2} & \text{if } \frac{1}{2} < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Verify that this can be the density of a random variable.

   (b) Find the cumulative distribution function $F(y)$.

   (c) Find the median of Y, i.e. a number c such that $P(Y \leq c) = P(Y \geq c) = \frac{1}{2}$.

   (d) Find the expected value $E(Y)$ and the variance $V(Y)$.

4. Solve the following questions, giving reasons for your answer.

   (a) A and B are two independent events. Are A^c and B^c independent events ?

   (b) Let X be a Bernoulli(p) random variable. Then X^2 is distributed as a Bernoulli(p) random variable. True or false.

   (c) Let X be a Geometric(p) random variable and Y be an independent Geometric(p) random variable. Then $X + Y$ is also a Geometric(p) random variable. True or false.

5. In answering a multiple choice exam question, a student either knows the correct answer or randomly guesses 1 of m alternatives. Let p be the probability that the students knows the answer. If the student gets the correct answer, then what is the probability that the student actually knew the answer?

1. Solution of problem 1.

   (a) Let L be the number of left handed children in a class of 20. Then

   $$P(L \geq 2) = 1 - P(L = 0) - P(L = 1) = 1 - (0.9)^{20} - 20C1(0.9)^{19}(0.1) = 0.608$$

   and $P(L = 0) = (0.9)^{20} = 0.1216$.

   (b) The probability that the first left–handed child will be found in the fourth class is

   $$P(L = 0)^3(1 - P(L = 0)) = (0.1216)^3(1 - 0.1216) = 0.001579$$

2. Solution of Problem 2.

   (a) The probability that one cookie has at least one raisin is $1 - e^{-3} = 0.9502$ The probability that both cookies contain at least one raisin is $\boxed{(1 - e^{-3})^2 = 0.9029}$

   (b) The probability that both cookies have no raisins is $(e^{-3})^2 = e^{-6}$. The probability that the two cookies have one raisin between them is $2(e^{-3})(e^{-3}\frac{3^1}{1!}) = 6e^{-6}$. So, the probability that the two cookies will have a total of at least two raisins is

   $$1 - e^{-6} - 6e^{-6} = 1 - 7e^{-6} = \boxed{0.9826}$$

3. Solution of problem 3.

   $$\int_{1/2}^{y} \frac{1}{t^2}\, dt = [-\frac{1}{t}]_{1/2}^{y} = 2 - \frac{1}{y}$$

   (a) $f_Y(y)$ is a function which is never negative and which has integral $\int f_Y(y)\, dy = \int_{1/2}^{1} \frac{1}{t^2}\, dt = 2 - \frac{1}{1} = 1$. These are the two requirements for being the density of a random variable.

   (b) $F_Y(y) = \begin{cases} 0 & \text{if } y \leq \frac{1}{2} \\ 2 - \frac{1}{y} & \text{if } \frac{1}{2} \leq y \leq 1 \\ 1 & \text{if } y \geq 1 \end{cases}$

   (c) $P(Y \leq c) = 2 - \frac{1}{c} = \frac{1}{2} \iff \boxed{c = \frac{2}{3}}$

   (d)

   $$\begin{aligned} E(Y) &= \textstyle\int_{1/2}^{1} y\frac{1}{y^2}\, dy = \int_{1/2}^{1} \frac{1}{y}\, dy = [\,\ln y]_{1/2}^{1} = \boxed{\ln 2 = 0.6931} \\ E(Y^2) &= \textstyle\int_{1/2}^{1} y^2\frac{1}{y^2}\, dy = \int_{1/2}^{1} dy = \frac{1}{2} \\ V(Y) &= E(Y^2) - E(Y)^2 = \boxed{\frac{1}{2} - \ln^2 2 = 0.0195} \end{aligned}$$

4. Solution of problem 4.

   (a) Yes. Reason: $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(AUB) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - [P(A) + P(B) - P(A)P(B)] = 1 - P(A) - P(B) + P(A)P(B)$. Now

   $$P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) = 1 - P(B) - P(A) + P(A)P(B).$$

   Hence $P(A^c \cap B^c) = P(A^c)P(B^c)$. Therefore they are independent.

(b) True. Reason: Since X takes values 0 or 1, X^2 takes values 0 or 1. Let us try to calculate the probability mass function of X^2. Let us call it $pmf_{X^2}(\cdot)$.

$$pmf_{X^2}(1) = P(X^2 = 1) = P(X = 1) = p$$

$$pmf_{X^2}(0) = P(X^2 = 0) = P(X = 0) = 1 - p$$

As the probability mass function solely identifies the random variable, we can conclude that X^2 is a Bernoulli(p) random variable.

(c) False. Reason:

First method: X measures the number of trials required to obtain the first success in a sequence of independent Bernoulli Trials. So does Y and independent of X. Therefore $X+Y$ measures the number of trials required to obtain the second success in a sequence of independent Bernoulli trials. Hence $X+Y$ is a negative binomial random variable with parameters (2,p) and NOT geometric.

Second method: X is a Geometric(p) random variable and thereby takes values $1, 2, 3, 4, 5, \ldots.$ Y is independent of X. Y is also a Geometric(p) random variable and thereby takes values $1, 2, 3, 4, 5, \ldots.$ Now, $X+Y$ takes values $2, 3, 4, 5, 6 \ldots.$ Let their probability mass function be denoted as pmf_X, pmf_Y and pmf_{X+Y}. We calculate the pmf_{X+Y}. For $k \geq 2$,

$$\begin{aligned}
pmf_{X+Y}(k) &= P(X+Y=k) \\
&= \sum_{m=1}^{k-1} P(X=m, Y=k-m) \\
&= \sum_{m=1}^{k-1} P(X=m)P(Y=k-m)(why?) \\
&= \sum_{m=1}^{k-1} pmf_X(m) pmf_Y(k-m) \\
&= \sum_{m=1}^{k-1} (1-p)^{m-1} p (1-p)^{k-m-1} p \\
&= \sum_{m=1}^{k-1} (1-p)^{k-2} p^2 \\
&= (1-p)^{k-2} p^2 \sum_{m=1}^{k-1} \\
&= (1-p)^{k-2} p^2 (k-1)
\end{aligned}$$

Hence $X+Y$ is not geometric(p). But one can realise that the pmf_{X+Y} resembles that of negative binomial (2,p). (the question did not ask that).

5. Solution of problem 5. Let

$$\begin{aligned}
K &= \text{the event that the student knows the correct answer} \\
C &= \text{the event that the student picks the correct answer}
\end{aligned}$$

Then

$$P(K|C) = \frac{P(KC)}{P(C)} = \frac{P(K)}{P(KC) + P(\overline{K}C)} = \frac{P(K)}{P(K) + P(\overline{K})P(C|\overline{K})} = \frac{p}{p + (1-p)(1/m)} = \boxed{\frac{mp}{(m-1)p+1}}$$