1) (Scheaffer p275# 6.5) The joint distribution of amount of pollutant emitted from a smokestack without a cleaning device (X_1) and with a cleaning device (X_2) is given by

$$f(x_1, x_2) = \begin{cases} k & \text{if } 0 \le x_1 \le 2,\ 0 \le x_2 \le 1 \text{ and } 2x_2 \le x_1 \\ 0 & \text{otherwise} \end{cases}$$

The reduction in amount of pollutant emitted due to the cleaning device is given by $U = X_1 - X_2$.

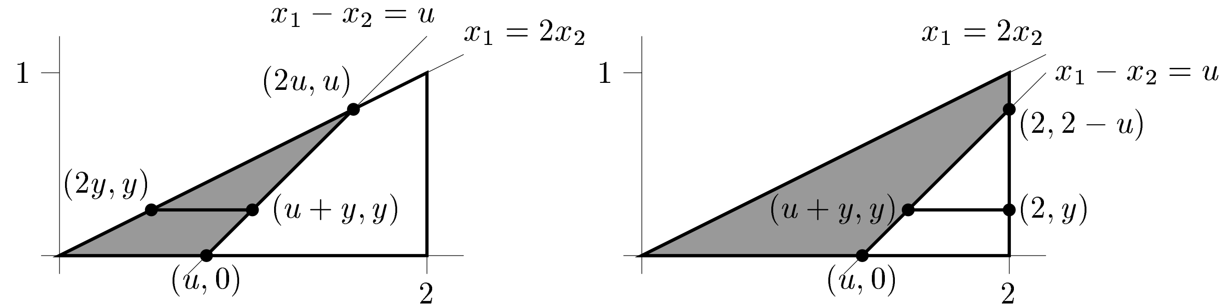
a) Find the probability density function for U.

b) Find $E(U)$.

Solution. a) If $u < 0$, $P(X_1 - X_2 \le u) = F_U(u) = 0$. If $u > 2$, $P(X_1 - X_2 \ge u) = 1 - F_U(u) = 0$. For $0 \le u \le 2$,

$$F_U(u) = P(X_1 - X_2 \le u) = \iint_{x_1 - x_2 \le u} f(x_1, x_2)\ dx_1 dx_2$$

The domain of integration for this integral is illustrated in the figures below. The figure on the left is valid for $0 \le u \le 1$ and the figure on the right is valid for $1 \le u \le 2$.



Furthermore, since the region $0 \le x_1 \le 2$, $0 \le x_2 \le 1$, $2x_2 \le x_1$ is a triangle of base 2, height 1 and area 1, $k = 1$. So, for $0 \le u \le 1$,

$$F_U(u) = P(X_1 - X_2 \le u) = \int_0^u dy \int_{2y}^{u+y} dx = \int_0^u dy\ (u - y) = -\tfrac{1}{2}(u - y)^2 \Big|_{y=0}^{y=u} = \tfrac{1}{2}u^2$$

For $1 \le u \le 2$,

$$F_U(u) = 1 - P(X_1 - X_2 \ge u) = 1 - \int_0^{2-u} dy \int_{y+u}^{2} dx = 1 - \int_0^{2-u} dy\ (2 - u - y)$$
$$= 1 - \tfrac{1}{2}(2 - u - y)^2 \Big|_{y=0}^{y=2-u} = 1 - \tfrac{1}{2}(2 - u)^2$$

Differentiating

$$f_U(u) = \tfrac{d}{du} F_U(u) = \boxed{\begin{cases} u & \text{if } 0 \le u \le 1 \\ 2 - u & \text{if } 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}}$$

b)

$$E(U) = \int u f_U(u)\ du = \int_0^1 u^2\ du + \int_1^2 u(2 - u)\ du = \tfrac{1}{3}u^3 \Big|_{u=0}^{u=1} + \left[u^2 - \tfrac{1}{3}u^3\right]_{u=1}^{u=2}$$
$$= \tfrac{1}{3} + 4 - \tfrac{8}{3} - 1 + \tfrac{1}{3} = \boxed{1}$$