(1) Solved in class.

(2a) Let X be the diameter of the pin chosen at random and let $Z = \frac{X-0.5}{0.006}$ Then Z has a standard normal distribution and

$$P(0.4925 < X < 0.5105) = P(\frac{0.4925-0.5}{0.006} < Z < \frac{0.5105-0.5}{0.006}) = 0.8544$$

(2a) Using Binomial distribution with p = 0.8544 and n= 3 the probability is

$$3C2(0.8544)^2(1-0.8544)^1 + 3C3(0.8544)^3(1-0.8544)^0 = 0.9426.$$

(3) Let X and Y be the distances of the two cuts from one end of the piece of lumber. The joint probability density of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{\ell^2} & \text{if } 0 \le x, y \le \ell \\ 0 & \text{otherwise} \end{cases}$$

The length of the middle piece is $|X-Y|$ and

$$\begin{aligned} E(|X-Y|) &= \tfrac{1}{\ell^2}\int_0^\ell dx \int_0^\ell dy\ |x-y| = \tfrac{2}{\ell^2}\int_0^\ell dx \int_0^x dy\ (x-y) = \tfrac{2}{\ell^2}\int_0^\ell dx\ \tfrac{x^2}{2} \\ &= \tfrac{2}{\ell^2}\tfrac{\ell^3}{6} = \boxed{\tfrac{\ell}{3}} \end{aligned}$$

(4a)

$$\begin{aligned} 1 &= \textstyle\int_0^\infty dy \int_{-y}^y dx\ c(y^2-x^2)e^{-y} = 2\int_0^\infty dy \int_0^y dx\ c(y^2-x^2)e^{-y} \\ &= \textstyle 2\int_0^\infty dy\ c(y^3 - \frac{1}{3}y^3)e^{-y} = \frac{4}{3}c\int_0^\infty dy\ y^3 e^{-y} = \frac{4}{3}c\Gamma(4) = \frac{4}{3}3!c = 8, c = \frac{1}{8} \end{aligned}$$

(4b)

$$f_Y(y) = \int_{-y}^{y} dx\ \frac{1}{8}(y^2-x^2)e^{-y} = \frac{1}{4}\int_0^y dx\ (y^2-x^2)e^{-y} = \frac{1}{4}[y^3 - \frac{1}{3}y^3]e^{-y} = \boxed{\tfrac{1}{6}y^3e^{-y}}$$

The condition $-y \le x \le y$ is equivalent to the condition $|x| \le y$. Hence the conditions $-y \le x \le y$, $0 < y < \infty$ are equivalent to the conditions $y \ge |x|$, $-\infty < x < \infty$ and

$$f_X(x) = \int_{|x|}^{\infty} dy\ \frac{1}{8}(y^2-x^2)e^{-y} = \frac{1}{8}\Big[(-y^2-2y-2+x^2)e^{-y}\Big]_{|x|}^{\infty} = \boxed{\tfrac{1}{4}(|x|+1)e^{-|x|}}$$

(4c)

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\ dx = \frac{1}{4}\int_{-\infty}^{\infty} x(|x|+1)e^{-|x|}(x)\ dx = \boxed{0}$$

since the integrand is odd under $x \to -x$ and the domain of integration is symmetric about $x = 0$.

[1] **Office hours:** Monday: 2:00-3:00 pm, Tuesday:10:30-11:30, 3:30-4:30 pm, Friday: 1:30-2:30pm

(4d)

$$
\begin{aligned}
P(X < \frac{1}{2}Y) &= \int_0^\infty dy \int_{-y}^{y/2} dx\ \tfrac{1}{8}(y^2 - x^2)e^{-y} = \tfrac{1}{8}\int_0^\infty dy[y^2x - \tfrac{1}{3}x^3]_{-y}^{y/2} e^{-y} \\
&= \tfrac{1}{8}\int_0^\infty dy[\tfrac{1}{2} - \tfrac{1}{3}\tfrac{1}{8} + 1 - \tfrac{1}{3}]y^3e^{-y} = \tfrac{9}{64}\int_0^\infty dy\ y^3e^{-y} = \tfrac{9}{64}\Gamma(4) = \tfrac{9}{64}3! \\
&= \boxed{\tfrac{27}{32} = 0.8438}
\end{aligned}
$$

(5a) Let X be a Bernoulli(1) random random variable. Then $pmf_X(1) = 1$, $pmf_X(0) = 0$, and $E(X) = 1$. Let Y=-X. Then Y assumes value -1 with probability 1. Therefore $pmf_Y(-1) = 1$,$pmf_Y(0) = 0$, and $E(Y) = -1$. Let $Z = XY$ then Z assumes value -1 with probability 1. Therefore $pmf_Z(-1) = 1$,$pmf_Z(0) = 0$, and $E(XY) = E(Z) = -1$. This implies that E(XY) = E(X)E(Y). But given X, Y is determined . Hence Y and X are clearly not independent.

(5b) done in class. Use distribution function method.

(5c)

$$
\begin{aligned}
P(X \le x, Y \le y) &= P(X \le x)P(Y \le y) \\
\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)dsdt &= \int_{-\infty}^{x} f_X(s)ds \int_{-\infty}^{y} f_Y(t)dt \\
&\quad \text{Differentiate partially both sides w.r.t. y first.} \\
\int_{-\infty}^{\infty} f(s,y)ds &= \int_{-\infty}^{x} f_X(s)ds f_Y(y) \\
&\quad \text{Differentiate partially both sides w.r.t. x first.} \\
f(x,y) &= f_X(x)f_Y(y)
\end{aligned}
$$

(6) Recognize that if you have two 5's and two 6's. Then there is one open slot left. That slot could have any number between one and six. Now proceed using multinomial distribution.

(7) When $Y = 5$ an ace was drawn of draw 5, but not on draws 1 through 4. Hence

$$
P(\text{king on draw } n|Y = 5) = \begin{cases} \frac{4}{48} & \text{if } n = 1, 2, 3, 4 \\ 0 & \text{if } n = 5 \\ \frac{4}{52} & \text{if } n > 5 \end{cases}
$$

so that

$$
P(X = n|Y = 5) = \begin{cases} (\frac{44}{48})^{n-1}\frac{4}{48} & \text{if } n = 1, 2, 3, 4 \\ 0 & \text{if } n = 5 \\ (\frac{44}{48})^4(\frac{48}{52})^{n-6}\frac{4}{52} & \text{if } n > 5 \end{cases}
$$

For example, when $n > 5$, in order to have $X = n$ you need a nonking on each of draws 1 through 4 (each with probability $\frac{44}{48}$), a nonking on draw 5 (which is automatic, since you got an ace on draw 5), a nonking on each of draws 6 through $n-1$ (each with probability $\frac{48}{52}$) and a king on draw n (with probability $\frac{4}{52}$). Hence

$$
\begin{aligned}
E(X|Y = 5) &= \textstyle\sum_{n=1}^{4} n(\frac{44}{48})^{n-1}\frac{4}{48} + \sum_{n=6}^{\infty} n(\frac{44}{48})^4(\frac{48}{52})^{n-6}\frac{4}{52} \\
&= \textstyle\sum_{n=1}^{4} n(\frac{44}{48})^{n-1}\frac{4}{48} + \sum_{m=0}^{\infty}(m+6)(\frac{44}{48})^4(\frac{48}{52})^{m}\frac{4}{52}
\end{aligned}
$$

But

$$\sum_{m=0}^{\infty}(m+6)r^m \quad = \textstyle\sum_{m=0}^{\infty}\left(5r^m + \frac{d}{dr}r^{m+1}\right) = \frac{5}{1-r} + \frac{d}{dr}\frac{r}{1-r} = \frac{5}{1-r} + \frac{1}{(1-r)^2}$$

so

$$\begin{aligned} E(X|Y=5) \quad &= \tfrac{4}{48} + 2(\tfrac{44}{48})\tfrac{4}{48} + 3(\tfrac{44}{48})^2\tfrac{4}{48} + 4(\tfrac{44}{48})^3\tfrac{4}{48} + (\tfrac{44}{48})^4\tfrac{4}{52}(\tfrac{5}{1-48/52} + \tfrac{1}{(1-48/52)^2}) \\ &= \tfrac{4}{48} + 2(\tfrac{44}{48})\tfrac{4}{48} + 3(\tfrac{44}{48})^2\tfrac{4}{48} + 4(\tfrac{44}{48})^3\tfrac{4}{48} + (\tfrac{44}{48})^4\tfrac{4}{52}(\tfrac{5\times 52}{4} + \tfrac{52^2}{4^2}) \\ &= \tfrac{1}{12} + 2(\tfrac{11}{12})\tfrac{1}{12} + 3(\tfrac{11}{12})^2\tfrac{1}{12} + 4(\tfrac{11}{12})^3\tfrac{1}{12} + 5(\tfrac{11}{12})^4 + \tfrac{52}{4}(\tfrac{11}{12})^4 \\ &= \tfrac{1}{12} + \tfrac{1}{6}(\tfrac{11}{12}) + \tfrac{1}{4}(\tfrac{11}{12})^2 + \tfrac{1}{3}(\tfrac{11}{12})^3 + 18(\tfrac{11}{12})^4 = \boxed{13.41} \end{aligned}$$

(8) Let N be the total number of eggs in a nest and X the number of viable ones. Then if $N = n$, X has a binomial distribution with number of trials n and probability p of success for each trial. Thus, if $N = n$, X has mean np and variance $np(1-p)$. That is

$$E(X|N=n) = np \qquad V(X|N=n) = np(1-p)$$

or

$$E(X|N) = pN \qquad V(X|N) = p(1-p)N$$

Hence

$$E(X) = E[E(X|N)] = E[pN] = pE[N] = \boxed{p\lambda}$$

and

$$V(X) = E[V(X|N)] + V[E(X|N)] = E[p(1-p)N] + V[pN] = p(1-p)E[N] + p^2V[N]$$

Recalling that N has a Poisson with mean λ, $E[N] = V[N] = \lambda$, so that

$$E(X) = \boxed{p\lambda} \qquad V(X) = p(1-p)\lambda + p^2\lambda = \boxed{p\lambda}$$