STRONG INDUCTION

Let $\{P(n) \mid n \in \mathbb{N}\}$ be a sequence of Mathematical statements

Assume

ⓐ $P(1)$ is true

ⓑ For $k \geq 2$, $P(i)$ is true for $i < k$ implies $P(k)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Follows from P.M.I

Take

$\varphi(n) = \{P(i) \text{ is true } \forall i \leq n\}$

Apply PMI to φ

Proof ① in Worksheet

Case 1: $|a| = |b|$

$\boxed{\text{Relatively prime}} \Rightarrow |a| = 1 = |b|$

[Explicitly write out Combination]

$a=1 \quad b=1 \qquad 2a - b = 1$

$a=-1 \quad b=1 \qquad a + 2b = 1$

$a=1 \quad b=-1 \qquad 2a + b = 1$

$a=-1, \quad b=-1 \qquad -2a + b = 1$

Case 2: $b = 0 \qquad (a,b)$ relatively prime $\quad |a| = 1$

$m = a, \quad n = 0$

to get $ma + nb = 1$

Case 3: $|a| > |b|$

W.L.O.G. $\quad a > 0 \;\&\; b > 0$

STRONG INDUCTION on $a+b$

$P(k) = \{ \exists\, m, n \quad ma + nb = 1 \mid a \,\&\, b$ are relatively prime, $a+b = k \}$

Assumptions :-

- $a+b=1 \implies P(1)$ is true

- Assume $a+b \geq 2$ $\quad P(l)$ is true for $l < a+b$

As $a > b$, $b > 0$, Consider b & $a-b$

b, a-b are relatively prime (*)
If $m|a$ & $m|a-b \implies m|b$.
As a, b are relatively prime $\implies$ (*)

Notice $l = b + a - b = a < a+b$. (*) As $P(l)$ is true we have $\exists\, m, n \in \mathbb{Z}$ such that $mb + n(a-b) = 1$.

$\implies na + (m-n)b = 1$

As $n \in \mathbb{Z}$ & $m-n \in \mathbb{Z} \implies P(a+b)$ is true.

$\therefore$ By strong Induction $P(n)$ is true for all $n \in \mathbb{N}$ $\square$

(2) in Worksheet $a, q, b \in \mathbb{Z}$

a, b are relatively prime

$a \mid qb \implies a \mid q$

Sol: From Question 1 [Because a and b are relatively prime]

$\exists\, m, n \in \mathbb{Z}$ st $ma + nb = 1$ — (**)

multiply both sides of (**) by q

$qma + qnb = q$ — (***)

$a \mid ab \Rightarrow \exists\ k \in \mathbb{Z}\quad ab = ka$

XXX $\Rightarrow$ $$a = qma + n(ka)$$
$$= a(qm + nk)$$ — Associative & Distributive properties

As $qm + nk \in \mathbb{Z} \Rightarrow a \mid a$ $\square$

(4) in Worksheet

Solution:- If $m \mid a$ & $m \mid b \Rightarrow m \mid a - kb \Rightarrow$
$m \mid a - kb$ & $m \mid b \Rightarrow m \mid a$

$\boxed{l \mid a \ \& \ l \mid b \iff l \mid a - kb \text{ and } l \mid b}$

Ex: $\gcd(a, b) = \gcd(a - kb, b)$ $\square$

(3) d is greatest common divisor of a & b, a, b are relatively prime

$S = \{ma + nb \mid m, n \in \mathbb{Z}\}$ $\qquad T = \{kd \mid k \in \mathbb{Z}\}$

$S \subseteq T$

$d \mid a$ & $d \mid b$ & $d = \gcd(a, b)$

$\Rightarrow a = kd$ & $b = ld$ for some $k, l \in \mathbb{Z}$ with k, l relatively prime

$\alpha \in S \Rightarrow \alpha = ma + nb \quad m, n \in \mathbb{Z}$

$\Rightarrow \alpha = m(kd) + n(ld)$

$= (mk + ln)d$

$mk + ln \in \mathbb{Z} \Rightarrow \alpha \in T$

$\beta \in T \Rightarrow \beta = nd$ for some $n \in \mathbb{Z}$

$\frac{a}{d}$ & $\frac{b}{d}$ are relatively prime integers

Apply Question 1 $\exists\, t, s \in \mathbb{Z}$

$t\frac{a}{d} + s\frac{b}{d} = 1 \Rightarrow ta + sb = d$

$\Rightarrow \underbrace{nta + nsb}_{\in S \text{ as } nt, ns \in \mathbb{Z}} = nd = \beta$ $\square$