Cardinality

If X and Y are nonempty sets, we define the expressions

$\mathrm{Card}(X) \leq \mathrm{Card}(Y)$, $\mathrm{Card}(X) = \mathrm{Card}(Y)$, $\mathrm{Card}(X) \geq \mathrm{Card}(Y)$

to mean:

$\exists \; f: X \to Y$ & injective $[f(x) = f(y) \Longrightarrow x = y]$

$\exists \; f: X \to Y$ & bijective [injective & surjective]

$\exists \; f: X \to Y$ & surjective $[\forall y \in Y \; \exists x \in X : f(x) = y]$

We also define

$\mathrm{Card}(X) < \mathrm{Card}(Y)$

to mean $\exists \; f: X \to Y$ injective but NO bijection

$\mathrm{Card}(X) > \mathrm{Card}(Y)$

$\exists \; f: X \to Y$ is surjective but there is no bijection

Convention: $\text{Card}(\phi) < \text{Card}(X) \quad X \neq \phi$

We will assume non-empty subsets

$\text{Card}(X) > \text{Card}(\phi)$

Proposition: $\text{Card}(X) \leq \text{Card}(Y)$

iff

$\text{Card}(Y) \geq \text{Card}(X)$

Proof:- let $card(X) \leq card(Y)$

this means

$\exists f : X \to Y$ & f is an injective function.

Consider $f(X) = \{y \in Y \mid \exists x \in X,\ f(x) = y\}$

[f is injective
So $\{x : f(x) = y\}$ is a singleton]

Define $g : Y \to X$

$$g(y) = \begin{cases} f^{-1}(y) & y \in f(X) \\ x_0 & y \notin f(X) \end{cases}$$

let $x_0 \in X$
(as $X \neq \phi$)

f is injective
So well defined

Ex: g is surjective

$\Longrightarrow \operatorname{card}(Y) \geq \operatorname{card}(X)$

Let $\mathrm{card}(Y) \geq \mathrm{card}(X)$

means $\exists\, g: Y \longrightarrow X$ & g is surjective.

$x \in X \Rightarrow g^{-1}(\{x\}) = \{y \in Y \mid g(y) = x\} \neq \phi$

$$\prod_{x \in X} g^{-1}(\{x\}) := \left\{ f: X \to \bigcup_{x \in X} g^{-1}(\{x\}) \;\middle|\; f(x) \in g^{-1}(\{x\}) \;\; \forall x \in X \right\}$$

Ex: $\prod_{x \in X} g^{-1}(\{x\}) \neq \phi \Rightarrow f \in \prod_{x \in X} g^{-1}(\{x\})$ is injective. $\square$