Question: Let $\{x_n\}_{n=1}^{\infty}$and $\{y_n\}_{n=1}^{\infty}$be a bounded sequence of real numbers. Let $\{z_n\}_{n=1}^{\infty}$be given by

$$z_n = x_n + y_n$$

for all $n \geq 1$. Show that

$$\limsup_{n\to\infty} z_n \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

Solution: Let $X = \limsup_{n\to\infty} x_n, Y = \limsup_{n\to\infty} y_n, Z = \limsup_{n\to\infty} z_n$.

As the sequences are bounded both X and Y are real numbers. Further $\{z_n\}_{n=1}^{\infty}$is also bounded and so Z is also a real number. We will show by contradiction that $Z \leq X + Y$.

Suppose $Z > X + Y$. Let $\epsilon = (Z - (X + Y))/2$. As Z is a limit point of $\{z_n\}_{n=1}^{\infty}$, there exist infinitely many p for which $|z_p - Z| < \epsilon$. That is for all $N \geq 1$ there exists $p \geq N$ such that

$$Z - \epsilon < z_p < z + \epsilon.$$

So for all $N \geq 1$ there exists $p \geq N$ such that

$$z_p > Z - \epsilon = (X + Y + Z)/2 \tag{1}$$

As $X = \limsup_{n\to\infty} x_n$, only a finite number of terms of $\{x_n\}_{n=1}^{\infty}$ are greater than $X + \epsilon/2$, so there exists $N_1 > 1$ such that for all $k \geq N_1$ we have

$$x_k < X + \epsilon/2;$$

and as $Y = \limsup_{n\to\infty} y_n$, only a finite number of terms of $\{y_n\}_{n=1}^{\infty}$ are greater than $Y + \epsilon/2$, so there exists $N_2 > 1$ such that for all $k \geq N_2$ we have

$$y_k < Y + \epsilon/2;$$

Therefore $N_3 = \max(N_1, N_2)$ we have for all $k \geq N_3$

$$z_k = x_k + y_k < X + Y + \epsilon = (X + Y + Z)/2. \tag{2}$$

Take $N = N_3 + 1$ in (1), to obtain a

$$p_0 \geq N_3 + 1 \text{ such that } z_{p_0} > (X + Y + Z)/2$$

$$\text{and as } p_0 > N_3 \text{ from (2), we have } z_{p_0} < (X + Y + Z)/2.$$

This is a contradiction. So

$$Z \leq X + Y.$$

$\square$