3 (a)

last class

-1 is a limit point

1 is a limit point

$$x_n = \begin{cases} \dfrac{n}{n+1} & \text{if } n\text{-even} \\ -\dfrac{n}{n+1} & \text{if } n\text{-odd} \end{cases} = \begin{cases} 1 - \dfrac{1}{n+1} & \text{if } n \text{ is even} \\ -1 + \dfrac{1}{n+1} & \text{if } n \text{ is odd} \end{cases}$$

$x \notin \{-1, 1\}$ it is not a limit point

- $x < -1$ let $\varepsilon_0 = \dfrac{|x-(-1)|}{2}$, $(x-\varepsilon_0, x+\varepsilon_0) \not\ni x_n \ \forall n \geq 1$

4 (b) $x_n = \begin{cases} 3 - \dfrac{1}{n+1} & n\text{-even} \\ -1 - \dfrac{1}{n+1} & n\text{-odd} \end{cases}$

As

$$\boxed{\frac{n}{n+1} = 1 - \frac{1}{n+1}}$$

Ex: imitate proof of 3(a) to finish proof that limit points of $\{x_n\}_{n\geq 1}$ are $\{3, -1\}$

4 (c) $x_0 \in \mathbb{R}$ & $x_n = \frac{1}{2} x_{n-1}$ $\forall n \geq 1$

Induction $\quad x_n = \dfrac{x_0}{2^n} \quad \forall\, n \geq 1$

$$= x_0 \left(\frac{1}{2}\right)^n$$

$\exists n \geq 0$ take $r = \frac{1}{2}$ and [illegible] in proof

$\left(\frac{1}{2}\right)^n \longrightarrow 0$ as $n \to \infty$.

[illegible] $\Rightarrow$ $x_0 \left(\frac{1}{2}\right)^n \longrightarrow 0$ as $n \to \infty$

$\Rightarrow x_n \to 0$ as $n \to \infty$

$$\boxed{\lim_{n \to \infty} x_n = 0}$$

$\therefore$ 0 is the only limit point of $\{x_n\}_{n \geq 1}$

largest limit point $= 0$

Smallest limit point $= 0$

$$\boxed{\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = 0}$$

$E =$ set of all limit points of $\{x_n\}_{n \geq 1} \subseteq \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

$\boxed{\text{$E$ is singleton and say $E = \{x\}$, $x \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$}}$ $\iff$ largest limit point (Exists) $=$ smallest limit point (Exists)

$\iff$ $x_n \to x$ as $n \to \infty$.

4(d) Exercise

(2) $\frac{x_{n+1}}{x_n}$ — limit points

(3) $(x_n)^{1/n}$ — limit points

$\boxed{\begin{array}{l}\text{Repeat (3) for sequence in (2)} \\ \qquad \text{(2) for sequence in (3)}\end{array}}$

Table

x_n	Smallest limit pt of $\frac{x_{n+1}}{x_n}$	Smallest limit pt of $(x_n)^{1/n}$	Largest limit pt of $(x_n)^{1/n}$	Largest limit pt of $\frac{x_{n+1}}{x_n}$

Example

strictly inequality across the row

$\Rightarrow (x-\varepsilon_0, x+\varepsilon_0)$ has no elements of the sequence $\{x_n\}_{n\geq 1}$

$\Rightarrow$ x is not a limit point

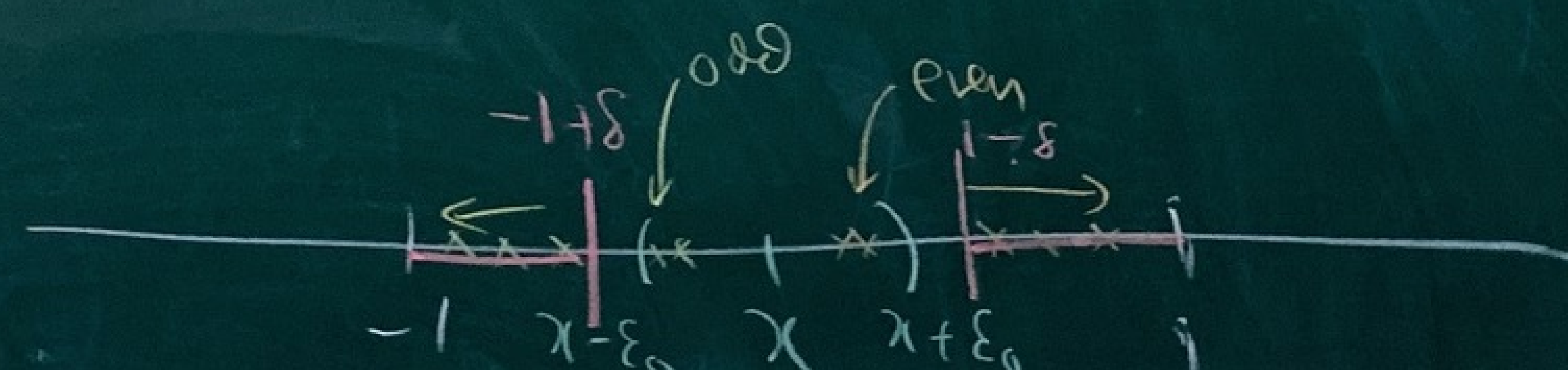
As $x<-1$ was arbitrary there are no limit points in $(-\infty,-1)$.

• $x>1$ $\quad \frac{|x-1|}{2} = \varepsilon_0 \quad (x-\varepsilon_0, x+\varepsilon_0)$ contains no elements of the sequence

$\Rightarrow$ x is not a limit point of $\{x_n\}_{n\geq 1}$.

$\therefore$ $x>1$ was arbitrary there are no limit points in $(1,\infty)$

• $-1<x<1$



Let $\varepsilon_0 = \frac{1}{2}\min\{|x-1|, |x-(-1)|\}$. Take $\delta = \frac{1}{2}\min\{|1-(x+\varepsilon_0)|, |-1-(x-\varepsilon_0)|\}$

Then $(-1,-1+\delta)\cap(x-\varepsilon_0, x)=\phi$ & $(1-\delta,1)\cap(x,x+\varepsilon_0)=\phi$ — (*)

By Archimedean principle $\exists\, N \geq 1$ such that

$$\frac{1}{N} < \delta$$

$\Rightarrow \forall\, n > N$

n-odd $\Rightarrow x_n = -1 + \frac{1}{n+1} < -1 + \delta$ $\left[\because n > N \Rightarrow \frac{1}{n+1} < \delta \right.$

n-even $\Rightarrow x_n = 1 - \frac{1}{n+1} > 1 - \delta$

$$\Rightarrow n > N \quad x_n \notin (-1+\delta,\ 1-\delta)$$

$\left[(x-\varepsilon_0, x+\varepsilon_0) \subseteq (-1,1) \quad (*) \right]$ and $(x-\varepsilon_0, x+\varepsilon_0) \subseteq (-1+\delta, 1-\delta)$, $\therefore n > N \quad x_n \notin (x-\varepsilon_0, x+\varepsilon_0)$.

$\Rightarrow (x-\varepsilon_0, x+\varepsilon_0)$ contains at most N elements. $\Rightarrow x$ is not a limit point of $\{x_n\}_{n \geq 1}$

3 (b) Find limit points

$$x_n = a^{1/n} \qquad 0 < a < 1$$

$$\Rightarrow 0 < x_n < 1 \qquad \left[\text{if not } a^{1/n} > 1 \Rightarrow a = \underbrace{a^{1/n} \cdots a^{1/n}}_{n\text{-times}} > 1 \right] *$$

$$x_n = \frac{1}{1+y_n} \qquad 0 < y_n$$

$$(1+y_n)^n = \sum_{k=0}^{n} {}^{n}C_k\, y^k \geq n y_n \qquad \boxed{n \geq 1}$$

$$\Rightarrow a^{1/n} = \frac{1}{1+y_n}$$

$$\Rightarrow a = \left(\frac{1}{1+y_n}\right)^n = \frac{1}{(1+y_n)^n} \leq \frac{1}{n y_n}$$

$$\Rightarrow 0 < y_n \le \frac{1}{an} \quad \forall\, n \ge 1.$$

let $\varepsilon > 0$ be given

Take $N > \frac{1}{a\varepsilon}$ $\quad N \in \mathbb{N}$

$\forall\, n \ge N \qquad 0 < y_n < \frac{1}{an} < \frac{1}{aN} < \varepsilon$

$\forall\, n \ge N \qquad 0 < y_n < \varepsilon$

$\Rightarrow \forall\, n \ge N \quad |y_n| < \varepsilon$

$$\boxed{\text{As } \varepsilon > 0 \text{ was arbitrary} \Rightarrow y_n \to 0 \text{ as } n \to \infty}$$

Algebra of limits

$\Rightarrow 1 + y_n \to 1$ as $n \to \infty$

$\Rightarrow \frac{1}{1+y_n} \to 1$ as $n \to \infty$

$\Rightarrow x_n \to 1$ as $n \to \infty$

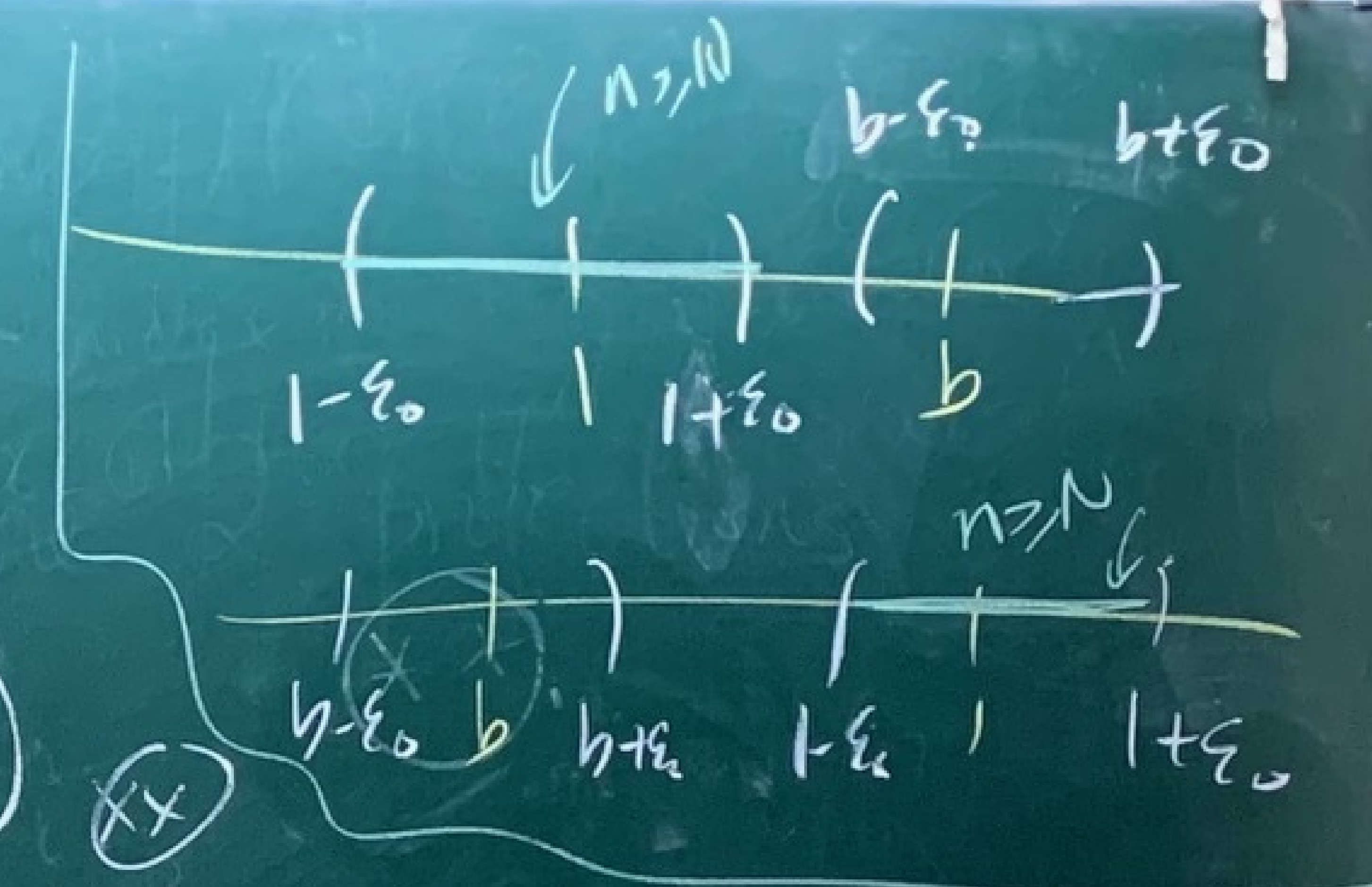
$\Rightarrow$
$\forall\, \varepsilon > 0\ \exists\, N \ge 1$ such that

$\forall\, n \ge N \quad |x_n - 1| < \varepsilon$ — (*)

ie $\forall\, n \ge N \quad x_n \in (1-\varepsilon, 1+\varepsilon)$

$\therefore$ 1 is a limit point $\{x_n\}_{n \ge 1}$

Take any $b \in \mathbb{R}$ $b \neq 1$ take $\epsilon_0 = \frac{|1-b|}{4}$



By (*) $\exists\ N \geq 1$ such that

$\forall n \geq N$ $x_n \in (1-\epsilon_0,\ 1+\epsilon_0)$ (**)

(**) $\Rightarrow$ $(b-\epsilon_0,\ b+\epsilon_0)$ has at most N elements

$\therefore$ b is not a limit point of $\{x_n\}_{n\geq 1}$

$\Rightarrow$ The set of limit points of $\{x_n\}_{n\geq 1} = \{1\}$.

3 (b) Find limit points

$0<r<1$; $x_n = \sum_{k=1}^{n} r^k \overset{\text{induction}}{=} \frac{r(1-r^n)}{1-r}$

Claim: $r^n \to 0$ as $n \to \infty$

Proof: $0 < r < 1 \Rightarrow r = \frac{1}{1+s}$ for some $s > 0$

$\Rightarrow \forall n \geq 1,\ y_n = r^n = \left(\frac{1}{1+s}\right)^n = \frac{1}{(1+s)^n} \leq \frac{1}{1+ns}$ [By induction]

$\Rightarrow 0 < y_n \leq \frac{1}{1+n\delta} \quad \forall n \geq 1.$

...t $\varepsilon > 0$ be given

...ake $N \in \mathbb{N}$ s.t. $N\delta > \frac{1}{\varepsilon}$

$\forall n \geq N \quad 0 < y_n < \frac{1}{1+n\delta} < \frac{1}{1+N\delta} < \frac{1}{N\delta} < \varepsilon$

$\forall n \geq N \quad 0 < y_n < \varepsilon$

$\Rightarrow \forall n \geq N \quad |y_n| < \varepsilon$

As $\varepsilon > 0$ was arbitrary $\Rightarrow y_n \to 0$ as $n \to \infty$.

Algebra of limits

$\Rightarrow 1 - y_n \to 1$ as $n \to \infty$

$\Rightarrow \frac{r}{1-r}(1-y_n) \to \frac{r}{1-r}\cdot 1$ as $n \to \infty$

$\Rightarrow x_n \to \frac{r}{1-r}$ as $n \to \infty$.

Ex.- Rest of argument is like (b) to show $r/_{1+r}$ is the only limit point

④ ⓐ $x_n = \begin{cases} 2 & \text{if } n\text{-even} \\ -2 & \text{if } n\text{-odd} \end{cases}$

[3ⓑ $x_n = \begin{cases} 1 - \frac{1}{n+1} & \text{for } n\text{-even} \\ -1 + \frac{1}{n+1} & \text{for } n\text{-odd} \end{cases}$]

Ex :- imitate 3ⓐ to get simpler proof that limit points of $\{x_n\}_{n\geq 1}$ are $\{2, -2\}$

$\limsup_{n\to\infty} x_n :=$ largest limit point $\equiv 2$

$\liminf_{n\to\infty} x_n :=$ Smallest limit point $\equiv -2$

$\boxed{\exists\, n_k \text{ subsequence st } x_{n_k} \to L \text{ as } k\to\infty}$

$\Updownarrow$

$L \in \mathbb{R}$ – limit point of $\{x_n\}$

- $\forall \varepsilon > 0$ $(L-\varepsilon, L+\varepsilon)$ has infinitely many elements of $\{x_n\}_{n\geq 1}$
- $L = \infty$ – limit point of $\{x_n\}$ if $\exists\, n_k$ subsequence st $x_{n_k} \to \infty$ as $k\to\infty$
- $L = -\infty$ – limit pt of $\{x_n\}$ if $\exists\, n_k$ subsequence st $x_{n_k} \to -\infty$ as $k\to\infty$