Quiz 9 solution

$\{x_n\}_{n\geq 1}$ & $\{y_n\}_{n\geq 1}$ — bounded sequence of real numbers.

$\{z_n\}_{n\geq 1}$

$z_n = x_n + y_n$ — is also a bounded sequence of real numbers

$\Big[\because \exists M_1 > 0 \quad |x_n| \leq M_1 \quad \forall n \geq 1$

$\exists M_2 > 0 \quad |y_n| \leq M_2 \quad \forall n \geq 1$

$\Rightarrow |z_n| \leq |x_n| + |y_n| \leq M_1 + M_2 \quad \forall n \geq 1 \Big]$

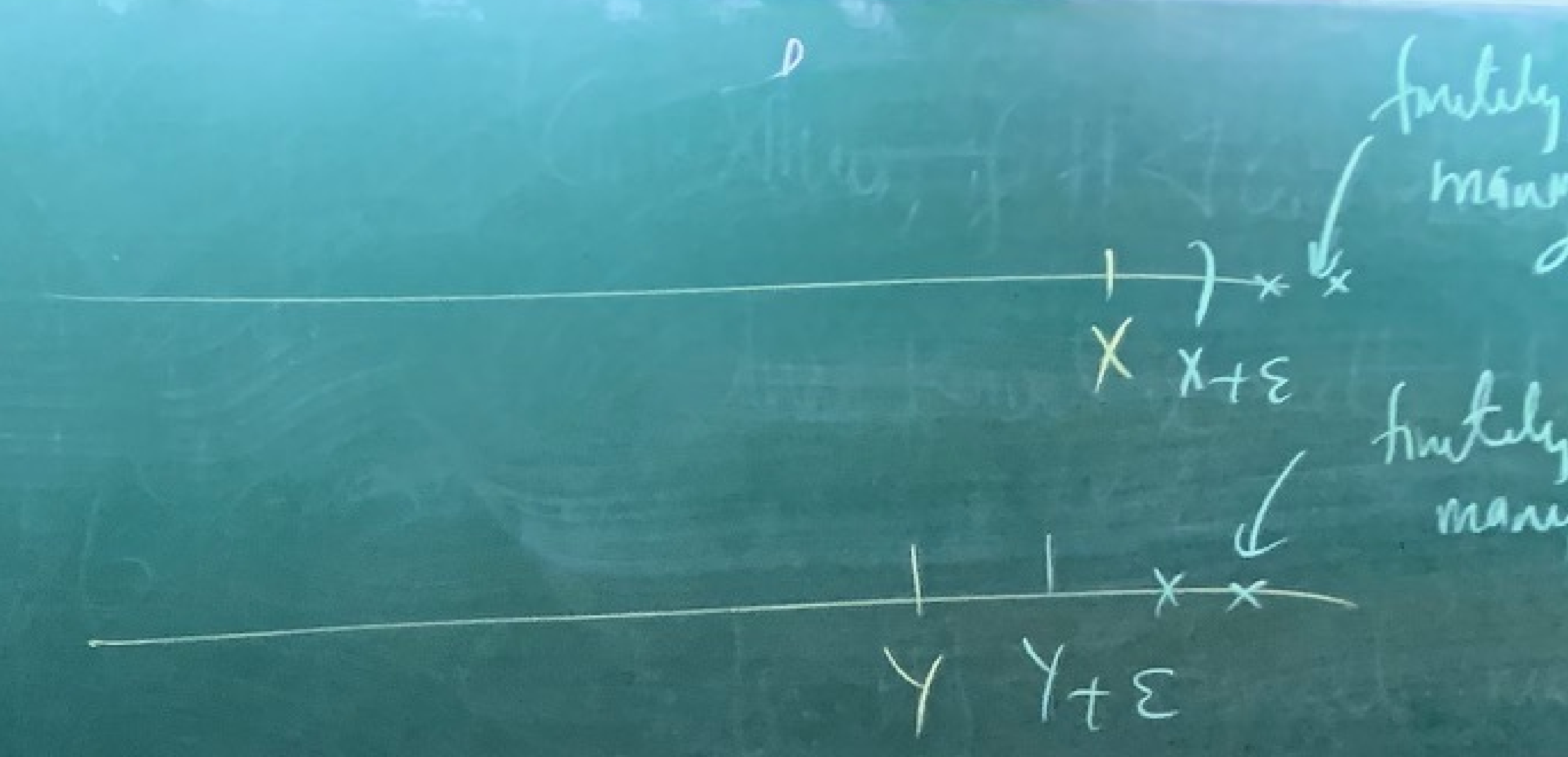
$\Rightarrow \limsup\limits_{n\to\infty} z_n = Z \in \mathbb{R}$, $\limsup\limits_{n\to\infty} x_n = X \in \mathbb{R}$

$\limsup\limits_{n\to\infty} y_n = Y \in \mathbb{R}$

let $\varepsilon > 0$ be given.

$\exists\, N_1 \geq 1$ $\quad x_n < X + \varepsilon \quad \forall\, n \geq N_1$

$\exists\, N_2 \geq 1$ $\quad y_n < Y + \varepsilon \quad \forall\, n \geq N_2$

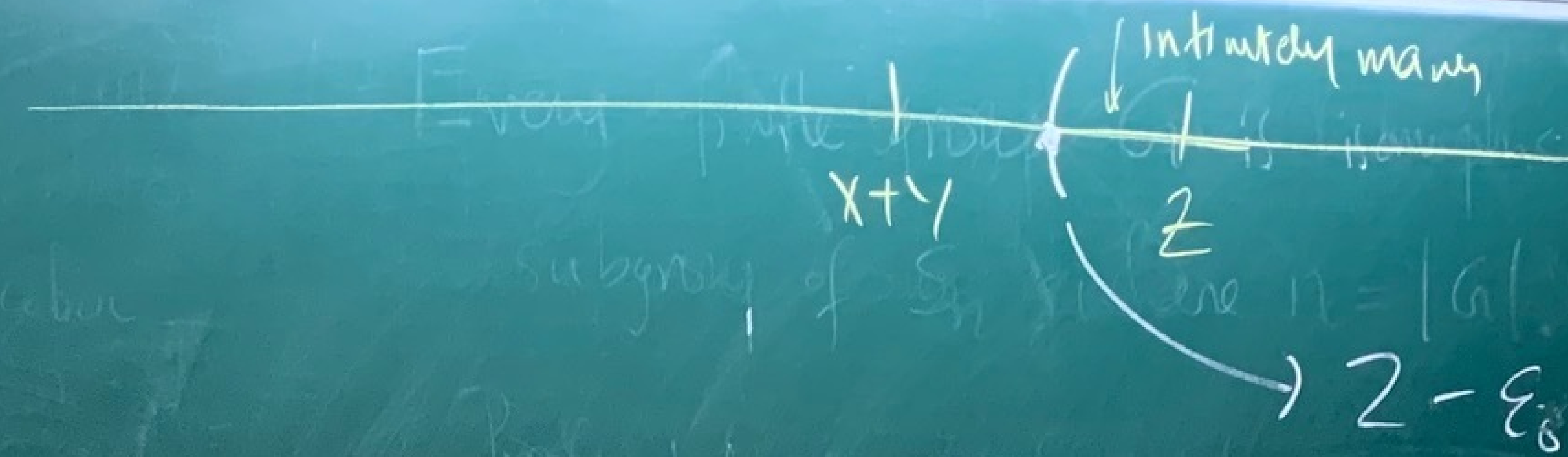


Take $N_3 = \max\{N_1, N_2\} + 1$

$n \geq N_3$ $\quad x_n + y_n < X + Y + 2\varepsilon$

$\Rightarrow \quad z_n < X + Y + 2\varepsilon \quad \forall\, n \geq N_3$ — (*)

Suppose $z > x+y$



let $\varepsilon_0 = \dfrac{z - x+y}{2} > 0$

$\varepsilon_0 = \dfrac{z-(x+y)}{2}$

$\forall\, m \geq 1 \quad \exists\, n \geq m$ such that

$$z_n > z - \varepsilon_0$$

$$= z - \frac{z-(x+y)}{2}$$

$z > x+y$

$$= \frac{z + x+y}{2} > x+y \quad — \quad X*$$

take $\varepsilon = \frac{\varepsilon_0}{4}$ in (*) to obtain N_3

such that $z_n < x + y + \frac{\varepsilon_0}{2}$ $\forall n \geq N_3$. — (+)

From (**) $\exists\, n_1 \geq N_3$ such that

$$z_{n_1} > z - \varepsilon_0 = x + y + \varepsilon_0 \quad \text{— (++)}$$

(+) & (++) yield a contradiction. $\square$

$\{x_n\}_{n\geq 1}$ $\{y_n\}_{n\geq 1}$ sequence of real number.

$z_n = x_n + y_n$

$$\limsup_{n\to\infty} z_n \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \quad — \circledast$$

$\circledast$ always true unless R.H.S of $\circledast$ is of the form "$\infty - \infty$"

Ex:- Show $z_n \to -\infty$ as $n\to\infty$ $\square$

Proof: $Z = \limsup_{n\to\infty} z_n$, $X = \limsup_{n\to\infty} x_n$, $Y = \limsup_{n\to\infty} y_n$

- $X, Y \in \bar{\mathbb{R}}$ Then Quiz 9 $\Rightarrow$ Result

- $X = \infty$. Then we know $Y \neq -\infty \Rightarrow X+Y = \infty$.

  As $z \leq \infty \Rightarrow z \leq X+Y$

- $X = -\infty$. Then we know $Y < \infty$.

  - $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n = X = -\infty \Rightarrow \liminf_{n\to\infty} x_n = -\infty$

$(x_n \to -\infty$ as $n\to\infty)$ $\forall M > 0$ $\exists N \geq 1$ such that $n \geq N$ $x_n < -M$ I

$\exists M_2 > 0$ & $N_1 \geq 1$ such that $n \geq N_1$ $y_n < M_2$ II

two (cases) $\to [Y = -\infty$ & $Y \in \mathbb{R}]$

1) c) Claim 1: $+_n$ is a well-defined binary operation on $\mathbb{Z}_n$.

Proof: i) Given $\bar{a}, \bar{b} \in \mathbb{Z}_n$ $\in \mathbb{Z}_n$,

$a+b \equiv r \pmod{n}$ for some $0 \le r \le n-1$

$\therefore \overline{a+b} = \bar{r} \in \mathbb{Z}_n$. So, $+_n$ sends $\mathbb{Z}_n^2$ to $\mathbb{Z}_n$.

ii) Given $(\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2) \in \mathbb{Z}_n^2$, $\ni$

$(\bar{a}_1, \bar{b}_1) = (\bar{a}_2, \bar{b}_2) \Rightarrow \bar{a}_1 = \bar{a}_2$ and $\bar{b}_1 = \bar{b}_2$

$\Rightarrow a_1 \equiv a_2$ and $b_1 \equiv b_2$

$\Rightarrow a_1 + b_1 \equiv a_2 + b_2 \equiv r$ for some $0 \le r \le n-1$

$\Rightarrow \overline{a_1+b_1} = \overline{a_2+b_2} = \bar{r}$

$\therefore +_n : \mathbb{Z}_n^2 \to \mathbb{Z}_n$ is a function

Claim 4: Existence of inverse

Proof: Given $\bar{r} \in \mathbb{Z}_n$, $r + (n-r) \equiv (n-r) + r \equiv 0 \pmod{n}$

$\Rightarrow \overline{r + (n-r)} = \overline{(n-r) + r} = \bar{0}$

$\Rightarrow \bar{r} +_n \overline{(n-r)} = \overline{(n-r)} +_n \bar{r} = \bar{0}$

$\therefore \overline{n-r} \in \mathbb{Z}_n$ is the inverse of $\bar{r}$.

$\therefore (\mathbb{Z}_n, +_n)$ is a group.

Claim 5: $(\mathbb{Z}_n, +_n)$ is abelian

Proof: Given $\bar{r_1}, \bar{r_2} \in \mathbb{Z}_n$, $r_1 + r_2 = r_2 + r_1 \Rightarrow \overline{r_1 + r_2} = \overline{r_2 + r_1} \Rightarrow \bar{r_1} +_n \bar{r_2} = \bar{r_2} +_n \bar{r_1}$.

Claim 2: $+_n$ is associative.

Proof: Given $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_n$,

$$a+(b+c) = (a+b)+c$$
$$\Rightarrow \overline{a+(b+c)} = \overline{(a+b)+c}$$
$$\Rightarrow \bar{a} +_n \overline{(b+c)} = \overline{(a+b)} +_n \bar{c}$$
$$\Rightarrow \bar{a} +_n (\bar{b} +_n \bar{c}) = (\bar{a} +_n \bar{b}) +_n \bar{c}$$

Claim 3: Identity element exists.

Proof: Given $0 \le r \le n-1$,

$$0 + r = r + 0 = r$$
$$\Rightarrow \overline{0+r} = \overline{r+0} = \bar{r}$$
$$\Rightarrow \bar{0} +_n \bar{r} = \bar{r} +_n \bar{0} = \bar{r} \Rightarrow \bar{0} \text{ is the identity element of } \mathbb{Z}_n.$$