Quiz

$\sup(x_n : n \geq 1) = 4$

$\inf(x_n : n \geq 1) = -6.5$

$$x_n = \begin{cases} 3 + \frac{1}{n} & n - \text{odd} \\ -6 - \frac{1}{n} & n - \text{even} \end{cases}$$

1. $E = \boxed{\{3, -6\}}$

2. $S_k = \sup\{x_n \mid n \geq k\} = \boxed{\begin{cases} 3 + \frac{1}{k} & k - \text{odd} \\ 3 + \frac{1}{k+1} & k - \text{even} \end{cases}}$, $\lim_{k \to \infty} S_k = \boxed{3} = \max(E)$

3. $I_k = \inf\{x_n . n \geq k\} = \boxed{\begin{cases} -6 - \frac{1}{k} & k - \text{even} \\ -6 - \frac{1}{k+1} & k - \text{odd} \end{cases}}$

$$\lim_{k \to \infty} I_k = \boxed{-6} = \boxed{\min}(E)$$

4. $\varepsilon = 0.001$. (a) $N = \boxed{\frac{1}{\varepsilon} + \frac{10}{n}} = 1000 + \frac{10}{n} = 1010$

$\boxed{\text{As } x_n \leq 3 + \frac{1}{n} \quad \forall n \geq 1}$

(b) $m = \boxed{2M+1}$ $\quad \varepsilon \quad x_m = 3 + \frac{1}{m} \geq 3 > 3 - \varepsilon$, $m \geq M$

(i) For all $\boxed{\text{But Finitely}}$ many $n \in \mathbb{N}$, we have $x_n \boxed{<} 3+\varepsilon$

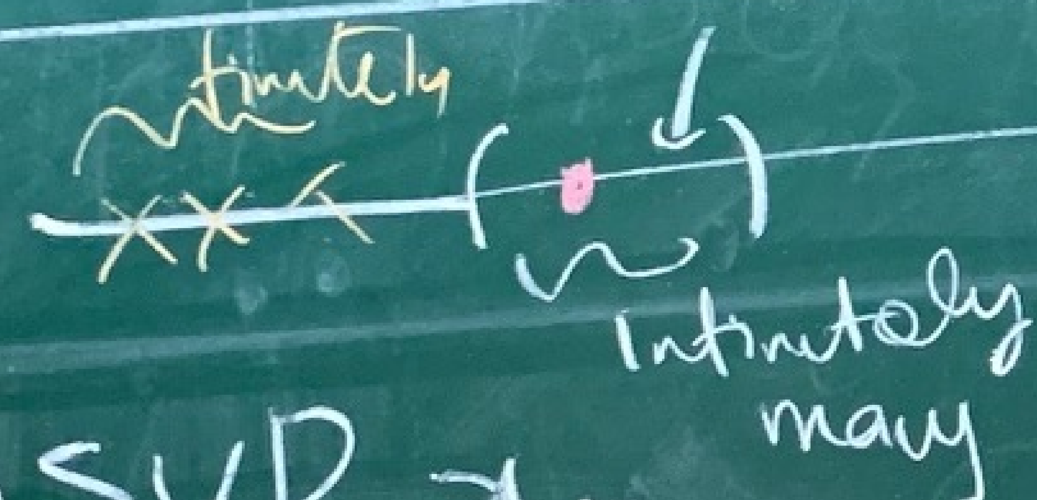
For $\boxed{\text{infinitely}}$ many $n \in \mathbb{N}$ we have $x_n \boxed{>} 3-\varepsilon$

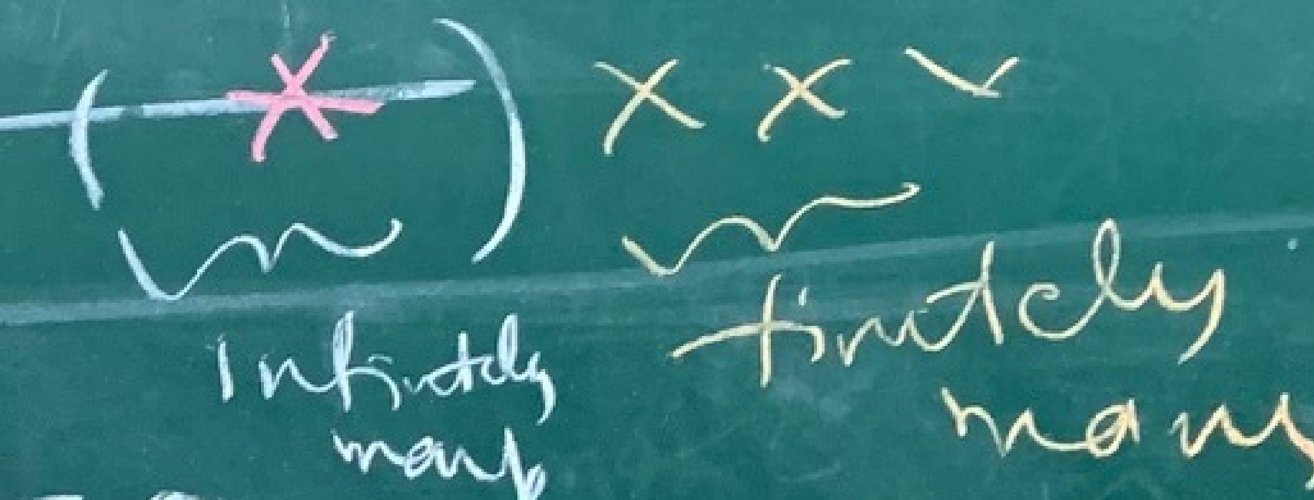
5. $\varepsilon = 0.005$ (a) $N = \boxed{\frac{1}{\varepsilon} + \frac{10}{=}} = 210$. As $x_n \geq -6 - \frac{1}{n} \quad \forall n \geq 1$.

(b) $m = \boxed{2M}$ $\quad x_m = -6 - \frac{1}{m} < -6 < -6 + \varepsilon$

(1) For $\boxed{\text{Infinitely}}$ many $n \in \mathbb{N}$ we have $x_n \boxed{<} -6+\varepsilon$

For all $\boxed{\text{But finitely}}$ many $n \in \mathbb{N}$ we have $x_n \boxed{>} -6-\varepsilon$





$\limsup\limits_{n\to\infty} x_n$	$\liminf\limits_{n\to\infty} x_n$
① $\alpha := \max(E)$, $E \equiv$ set of limit points	① $\beta := \min(E)$, E = set of limit points
② $k \geq 1$ $S_k = \sup\{x_n \mid n \geq k\}$ $S_k \to \max(E)$ as $k \to \infty$	② $k \geq 1$ $I_k = \inf\{x_n \mid n \geq k\}$ $I_k \to \min(E)$ as $k \to \infty$
③ $\forall \varepsilon > 0$ $\exists N : n \geq N$ $x_n < \alpha + \varepsilon$ ② $\forall N \geq 1$ $\exists n \geq N$ $x_n > \alpha - \varepsilon$	③ $\forall \varepsilon > 0$ $\exists N \geq 1$ $\forall n \geq N$ $x_n > \beta - \varepsilon$ ② $\forall N \geq 1$ $\exists n \geq N$ $x_n < \beta + \varepsilon$

$\alpha \in \mathbb{R}$

$x_n = \{\alpha n\}$

- $\alpha \in \mathbb{Z}$ $\quad x_n = 0 \quad \forall n \geq 1$
- $\alpha = p/q$ $\quad q \neq 0,$ $\quad$ limit points
  $p, q \in \mathbb{Z}$ $\quad \left\{ \frac{k}{q} \,\middle|\, k = 0, \dots, q-1 \right\}$
- $\alpha \in \mathbb{Q}^c$, $\quad$ limit points $= [0,1]$

HW8

$\frac{a_{n+1}}{a_n}$

$(a_n)^{\frac{1}{n}}$

Q 2 (d), (g)

2 (b) $x_n = \frac{2^n}{n!}$

Induction:

$n \geq 3$ $n! \geq 3^{n-2} \cdot 2 \cdot 1$

$n \geq 3$ $0 < x_n < \frac{2^n}{3^{n-2} \cdot 2}$

$\Rightarrow 0 < x_n < \left(\frac{2}{3}\right)^n \cdot \frac{9}{2}$ $\Bigg\} \Rightarrow$ $x_n \to 0$ as $n \to \infty$

$\boxed{0 < a < 1 \quad a^n \to 0 \text{ as } n \to \infty}$

$\boxed{\frac{x_{n+1}}{x_n} = \frac{2}{n+1} \leq \frac{2}{3} \quad n \geq 2}$

$\downarrow$

$x_{n+1} \leq \left(\frac{2}{3}\right)^n c_0$

$$0 < x_n < \left(\frac{2}{3}\right)^n \frac{9}{2}$$

$$S_n = \sum_{k=1}^{n} x_k < \frac{9}{2} \underbrace{\sum_{k=1}^{n} \left(\frac{2}{3}\right)^k}_{Z_n}$$

(*) $S_n < Z_n$

(**) $Z_n < Z_{n+1}$ & $Z_n \to 9$ as $n \to \infty$

$\Rightarrow Z_n < 9 \ \forall n \geq 1$

$$\Rightarrow \underbrace{S_n < 9 \ \forall n \geq 1}_{(+)}$$

$\boxed{x_k > 0}$ $\exists N$

$$\frac{x_{n+1}}{x_n} \le a < 1 \quad \forall n \geq N$$

$$\Big\}\ c\sum_{k=1}^{n} a^k$$

$$S_n = \sum_{k=1}^{n} x_k$$

$S_n \to \alpha \in \mathbb{R}$

as $n \to \infty$

$\exists N$

$$\frac{x_{n+1}}{x_n} \geq b > 1$$

$$\Big\}\ c\sum_{k=1}^{n} b^k$$

$$S_n = \sum_{k=1}^{n} x_k$$

$S_n \to \infty$ as $n \to \infty$

$\Rightarrow S_n < 9 \quad \forall n \geq 1$

$$\underbrace{\hphantom{xxxx}}_{(+)}$$

$$\left(1+\frac{1}{n}\right)^n \to e \text{ as } n \to \infty$$

$$\sum_{k=1}^{n} \frac{1}{k!} \le \sum_{k=1}^{n} \frac{e}{3^k}$$

$\lim_{n\to\infty}$

$x_k \geq 0 \quad \exists N : n \geq N$

$\Rightarrow (x_n)^{\frac{1}{n}} \leq a < 1$

$\Downarrow$

$S_n = \sum_{k=1}^{n} x_k$

$S_n \to \alpha$ as $n \to \infty$

$n \geq N$

$\Rightarrow (a_n)^{\frac{1}{n}} \geq b > 1$

$\Downarrow$

$S_n = \sum_{k=1}^{n} x_k$

$S_n \to \infty$ as $n \to \infty$

$x_n > 0$

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} (x_n)^{\frac{1}{n}} \leq \limsup_{n\to\infty} (x_n)^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$$

$$\boxed{\frac{x_{n+1}}{x_n} \leq M \quad \forall n \geq N \Rightarrow x_n \leq C M^n \quad \forall n \geq N}$$

$$\boxed{0 < r < 1 \quad \alpha \in \mathbb{R} \qquad \frac{r^n}{n^\alpha} \to 0 \text{ as } n \to \infty}$$

as $n \to \infty$

$\sum_{k=1}^{n} \leq \frac{}{3^k}$

$S_n = \sum_{k=1}^{n} x_k$ if $S_n \to a$ as $n \to \infty$ the series converges to a

$a \equiv \sum_{k=1}^{\infty} x_k$

- $\sum_{k=1}^{n} r^k = y_n$ $0<r<1$ $y_n \longrightarrow \frac{r}{1-r}$ as $n \to \infty$ (Done)

$r = \frac{2}{3}$ $\frac{9}{2} \sum_{k=1}^{n} \left(\frac{2}{3}\right)^k \longrightarrow \frac{9}{2} \cdot \frac{\frac{2}{3}}{1-2/3} = 9$ as $n \to \infty$

$S_n = \sum_{k=1}^{n} x_k$, $x_k \geq 0 \Rightarrow$ $S_n < S_{n+1}$ $n \geq 1$ claim: $S_n \leq a$ $\forall n \geq 1$

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