3 (a)

$$x_n = \frac{(-1)^n n}{n+1} \qquad L := \{\text{set of limit points of } \{x_n\}_{n\geq 1}\}$$

To show:- $L = \{-1, 1\}$

Proof: $$x_n = \begin{cases} \dfrac{-n}{n+1} & n = 2k-1 \text{ for } k \geq 1 \\ \dfrac{n}{n+1} & n = 2k \text{ for } k \geq 1 \end{cases}$$

$$= \begin{cases} -1 + \dfrac{1}{n+1} & n = 2k-1 \text{ for } k \geq 1 \\ 1 - \dfrac{1}{n+1} & n = 2k \text{ for } k \geq 1 \end{cases}$$

$x = -1$

let $\delta > 0$ be given

By archimedean property $\exists M \geq 1$ st $M > 1/\delta$.

observe if $n = 2k-1$ for $k \geq 1$

$$|x_n - (-1)| = \frac{1}{n+1}$$

let $N \geq 1$ be given

$n_0 = 2\max(N, M) + 1$

By definition

n_0 is odd, $n_0 > N$, $n_0 > M$ $\&$ ...

$n_0 \quad |x_{n_0} - (-1)| = \frac{1}{n_0+1} < \frac{1}{M+1} < \delta$

As $\delta > 0$ & $N \geq 1$ were arbitrary

we have shown

$\forall \delta > 0 \ \forall N \geq 1 \ \exists n_0 > N$ s.t. $|x_{n_0} - (-1)| < \delta$

This is implies -1 is a limit point of $\{x_n\}_{n \geq 1}$

To show

$x = 1$ is a limit point of $\{x_n\}_{n \geq 1}$

$x \notin \{-1, 1\}$ is not a limit point of $\{x_n\}_{n \geq 1}$

$$\lim_{n\to\infty} \frac{2-n}{1+n^2+n} = 0$$

Proof: For all $\varepsilon > 0$ by Archimedean Property on $\mathbb{R}$ there is a $N \in \mathbb{N}$ such that $N > 2-\varepsilon$.

① ×

For any $n > N$ we have

$$a_n \overset{①}{=} \frac{2-n}{1+n^2+n} \underset{(n>2\text{ fail})}{\overset{\times}{<}} \frac{2-n}{1} \le 2-N < \varepsilon$$

What is ε?

Thus for all $n \geq N$, $|a_n - 0| < \varepsilon$ for all $\varepsilon > 0$. Since

not show

ⓧ ⓧ ⓧ

$\varepsilon > 0$ was arbitrary, we are done. $\square$

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Proof does not make sense

NOT TRUE! | will imply $n \geq N$ Then $a_n = 0$

Qualifier in Coming Proof statement

Proof: Let $\varepsilon > 0$ be given.

By Archimedean property $\exists\, N \in \mathbb{N}$ such that $N > 3/\varepsilon$

Observe $|a_n| = \dfrac{|2-n|}{n^2+n+1}$

$$\left(\begin{array}{l}\text{As } n^2+n+1 > n^2 \ \forall n \geq 1 \\ |2-n| > 0\end{array}\right) \leq \frac{|2-n|}{n^2}$$

$$(\text{Triangle inequality}) \leq \frac{2}{n^2} + \frac{1}{n} \leq \frac{3}{n} \quad -(*)$$

(As $n^2 \geq n \ \forall n \geq 1$)

If $n > N$ then by (*)

$$|a_n| < \frac{3}{n} < \frac{3}{N} < \frac{3}{3/\varepsilon} = \varepsilon$$

$\therefore \ \forall n > N \quad |a_n| < \varepsilon$

Since $\varepsilon > 0$ was arbitrary, we have shown

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $\forall n > N \ \ |a_n| < \varepsilon$.

$\left[\therefore \lim_{n \to \infty} \frac{2-n}{n^2+n+1} = 0 \right]$

This proves the result $\square$

$$\lim_{n\to\infty} \frac{1+n}{1+n^2+n} = 0$$

$\varepsilon < \frac{1}{2}$
$\Rightarrow 2\varepsilon - 1 < 0$

Proof: let $\varepsilon > 0$ be given.

By Archimedean property on $\mathbb{R}$ there is a $N \in \mathbb{N}$ such that $\frac{1}{N} < 2\varepsilon - 1$

For any $n > N$

$$|a_n| = \frac{1+n}{1+n^2+n}$$

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$$< \frac{1+n}{2n}$$

$$= \frac{1}{2n} + \frac{1}{2}$$

$$\times \; = \frac{1}{2N} + \frac{1}{2} \quad <$$

Be careful

$$\times \; < \frac{1}{2}(2\varepsilon - 1) + \frac{1}{2} = \varepsilon$$

Thus for all $n \geq N$ $|a_n - 0| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary we are done Δ

Let $a_n = \frac{1+n}{n^2+n+1}$ $\forall n \geq 1$.

Proof: Let $\varepsilon > 0$ be given.

By Archimedean property $\exists N \in \mathbb{N}$ such that $N > 2/\varepsilon$

Observe $|a_n| = \frac{|1+n|}{n^2+n+1}$

$\left(\text{As } n^2+n+1 > n^2 \ \forall n \geq 1 \quad |1+n| > 0\right) \leq \frac{|1+n|}{n^2}$

$= \frac{1}{n^2} + \frac{n}{n^2} = \frac{1}{n^2} + \frac{1}{n} \leq \frac{2}{n}$ — $\times$

(As $n^2 \geq n$ $\forall n \geq 1$)

If $n > N$ then by (*)

$$|a_n| < \frac{2}{n} < \frac{2}{N} < \frac{2}{2/\varepsilon} = \varepsilon$$

$\therefore \forall n > N \quad |a_n| < \varepsilon$

Since $\varepsilon > 0$ was arbitrary, we have shown

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ st $\forall n > N \quad |a_n| < \varepsilon$. $\quad \left[\therefore \lim_{n \to \infty} \frac{1+n}{n^2+n+1} = 0 \right]$

This proves the result $\square$

2 $a \in \mathbb{R}$ is a limit point of $\{x_n\}_{n\geq 1}$

$\forall \delta > 0$ $(a-\delta, a+\delta)$

if $\boxed{\text{every interval around}}$ a

$\boxed{\text{containing infinitely many}}$ of $\{x_n\}_{n\geq 1}$

$\forall N \geq 1$ $\exists n_0 > N$

logical notation

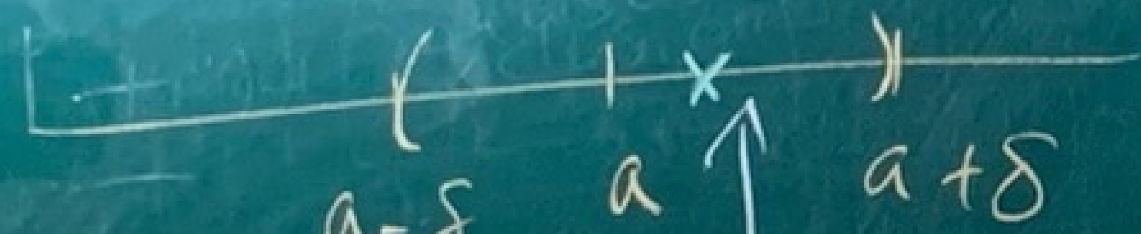
$a \in \mathbb{R}$ is a limit point of $\{x_n\}_{n\geq 1}$ if

$\forall \delta > 0, \ \forall N \geq 1 \ \exists n_0 > N$ $a_{n_0} \in (a-\delta, a+\delta)$

$\quad n_0 \equiv n_0(\delta, N)$

Picture of statement

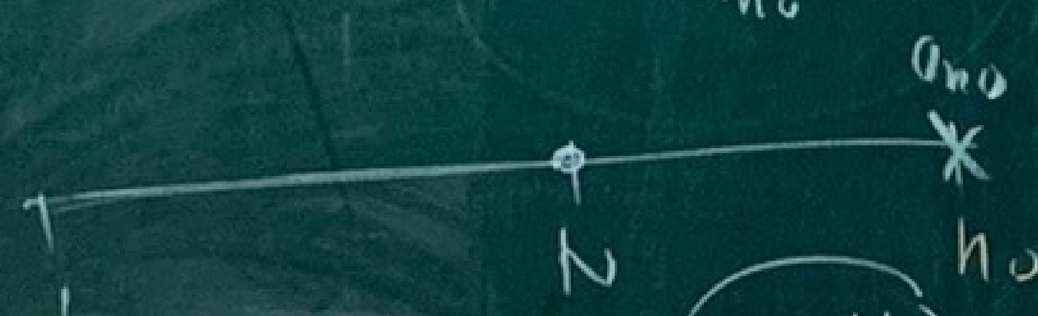
$\delta > 0$ arbitrary interval



(infinitely many)

$\forall N \geq 1 \quad \exists n_0 > N$

$a_{n_0} \in (a-\delta, a+\delta)$



$\exists \delta_0 > 0 \quad (a-\delta_0, a+\delta_0)$ contains only finitely many $\{a_n\}_{n\geq 1}$

$\Uparrow$

$\# \; a_n \in (a-\delta_0, a+\delta_0) \leq N_0$

$\Uparrow$

logical notation

$b \in \mathbb{R}$ is NOT a limit point of $\{x_n\}_{n\geq 1}$

$\exists \delta_0 > 0 \quad \exists N_0 = N(\delta_0) \quad \boxed{\forall n > N_0 \quad a_n \notin (a-\delta_0, a+\delta_0)}$