Quiz Solution

- Let set $A := \{x \in \mathbb{R} \mid 0 < x < 1\}$

  $B := \{1\}$

- $A = \left\{1 - \frac{1}{n} \;\middle|\; n \geq 1\right\}$

  $B = \{1\}$

- $A = [0,1] \cap \mathbb{Q}$

  $B = [0,1] \cap \mathbb{Q}^c$

- $A = \{p \in \mathbb{Q} \mid p^2 < 2\}$

  $B = \{p \in \mathbb{Q}^c \mid p^2 < 2\}$

(a) $a_n \to 0$ as $n \to \infty$

Short notation: $n_k < n_{k+1}$ $\forall k \geq 1$
$(a_{n_k})_{k\geq 1}$ – subsequence of $(a_n)_{n\geq 1}$

(c) Every subsequence $(a_n)_{n\geq 1}$ has a further subsequence that converges to 0.

Proof :- (a) $\Rightarrow$ (c)

Direct Proof

Asiume (a) holds
Show (c) holds

(a) holds
(*) $\forall \varepsilon > 0$ $\exists N \geq 1$ such that $\forall n \geq N$
$|a_n| < \varepsilon$

let $\{a_{n_k}\}_{k\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$

Denote $b_k = a_{n_k}$ $\forall k \geq 1$

sequence
$a : \mathbb{N} \to \mathbb{R}$

$g : \mathbb{N} \to \mathbb{N}$
$n < m \Rightarrow g(n) < g(m)$

$b = a \cdot g$
an subsequence

Observe: a subsequence of $\{a_{n_k}\}_{k\geq 1}$ (= $\{b_k\}_{k\geq 1}$) can be denoted $\{a_{n_{k_\ell}}\}_{\ell\geq 1}$ (= $\{b_{k_\ell}\}_{\ell\geq 1}$) $k_\ell < k_{\ell+1}\ \forall \ell \geq 1$

(+)

- $\{a_{n_k}\}_{k\geq 1}$ is a subsequence of $\{a_{n_k}\}_{k\geq 1}$ [$k_\ell = \ell$]

let $\varepsilon > 0$ be given.

By (*) $\exists N \geq 1$ such that $n \geq N \Rightarrow |a_n| < \varepsilon$ (xx)

Observe: $n_k \geq k$

Take $k > N \Rightarrow n_k > N \overset{(xx)}{\Rightarrow} |a_{n_k}| < \varepsilon$.

$k \geq N \Rightarrow |a_{n_k}| < \varepsilon$ or equivalently $k > N \Rightarrow |b_k| < \varepsilon$ (xxx)

As $\varepsilon > 0$ was arbitrary

(xxx) $\Rightarrow$ $b_k \to 0$ as $k \to \infty$.

So (-) holds by (+)

7 ⓒ There exists a subsequence of $\{a_n\}_{n\geq 1}$ for which there is no further subsequence that converges to 0.

ⓒ $\Rightarrow$ ⓐ

Proof by Contrapositive.

Assume ¬ⓐ ; show ¬ⓒ

¬ⓐ $\Rightarrow$ $\exists \varepsilon_0 > 0$ s.t.

$\forall N \geq 1$ $\exists n_1 \geq N$, $|a_{n_1}| \geq \varepsilon_0$

— ↯

take ε_0 as in (*)

$N=1 \Rightarrow \exists\, n_1 \geq 1 \quad |a_{n_1}| \geq \varepsilon_0$

$N_2 = n_1+1 \Rightarrow \exists\, n_2 \geq N_2 \quad |a_{n_2}| \geq \varepsilon_0 \quad [n_2 \geq N_2 = n_1+1 \Rightarrow n_2 > n_1]$

Suppose for $k \in \mathbb{N}$, $\exists\, n_1 < n_2 < \cdots < n_k$ such that $|a_{n_i}| \geq \varepsilon_0$.

$N_k = n_k + 1$ by (*) $\exists\, n_{k+1} \geq N_k$ s.t. $|a_{n_{k+1}}| \geq \varepsilon_0$

as $n_{k+1} \geq N_k \Rightarrow n_{k+1} > n_k$

$\therefore$ By induction $\exists \{n_k\}_{k \geq 1}$, $n_k < n_{k+1}$ s.t. $|a_{n_k}| \geq \varepsilon_0 \quad \forall k \geq 1$.

Ex: To finish the proof