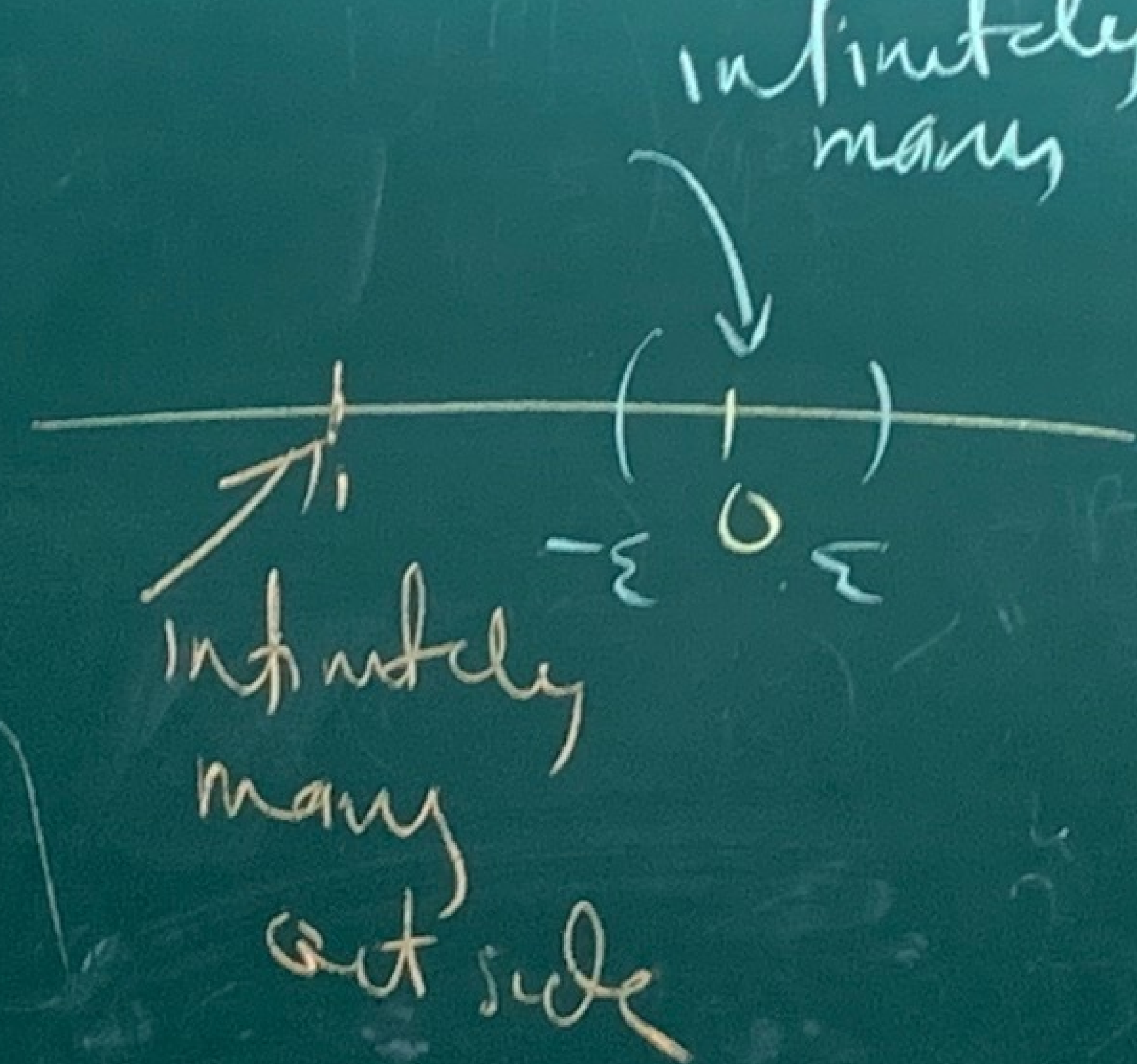
1 (a) (i) $a_n = 0 \quad \forall n \geq 1$

(ii) $a_n = \begin{cases} 0 & n = 2k \text{ for } k \geq 1 \\ 1 & n = 2k-1 \text{ for } k \geq 1 \end{cases}$



$\forall \varepsilon > 0 \quad \forall N \geq 1 \quad \exists n_0 \geq N \quad a_{n_0} \in (-\varepsilon, \varepsilon)$

"infinitely many elements of the sequence"

In Example (i)

$\{n \geq 1 \mid a_n = 0\} = \mathbb{N}$

$\mathbb{N}$ are infinite

let $\varepsilon > 0$ be given

$\{n \geq 1 \quad a_n \in (-\varepsilon, \varepsilon)\} = \mathbb{N} \Rightarrow$ there are infinitely many in $(-\varepsilon, \varepsilon)$

As $\varepsilon > 0$ was arbitrary we are done.

(ii) $a_n = \begin{cases} 0, & n = 2k \quad k \geq 1 \\ 1 & n = 2k-1 \quad k \geq 1 \end{cases}$

let $\varepsilon > 0$ be given.

let $N \geq 1$ be given

take $n_0 = 2N \Rightarrow a_{n_0} = 0 \in (-\varepsilon, \varepsilon)$

As $\varepsilon > 0$ and $N \geq 1$ were arbitrary there are infinitely

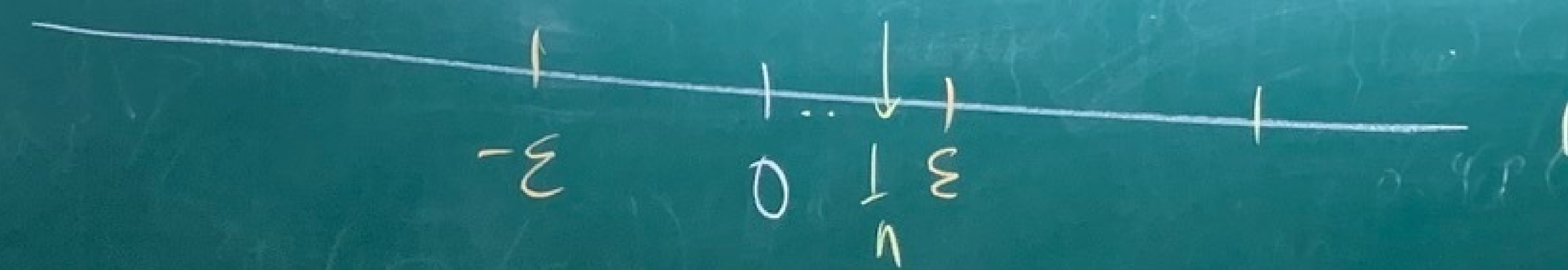
Many elements of the sequence $\{a_n\}_{n \geq 1}$ $\in (-\varepsilon, \varepsilon)$ $\forall \varepsilon > 0$.

1 (b) $\forall \varepsilon > 0$ all but finitely many are in $(-\varepsilon, \varepsilon)$

$\forall \varepsilon > 0\ \exists N \geq 1$ st $\forall n \geq N$ $a_n \in (-\varepsilon, \varepsilon)$

Example (iii)

$a_n = \frac{1}{n^\alpha}$ $\quad \alpha \in \mathbb{N}$



$\sqrt[n]{x}$ $\quad x^{\frac{1}{n}} = y$

$\Updownarrow$

$\exists! y > 0\ y^n = x$

let $\varepsilon > 0$ be given

let $N = \left(\left[\frac{1}{\varepsilon^{1/\alpha}} \right] + 1 \right)$

$n \geq N \implies \frac{1}{n^\alpha} < \varepsilon$

$\implies a_n \in (-\varepsilon, \varepsilon)$

As $\varepsilon > 0$ was arbitrary $\implies$

$\forall \varepsilon > 0$ all but finitely many are in $(-\varepsilon, \varepsilon)$.

let $\varepsilon > 0$ be given

(i) $\{ n \geq 1 \mid a_n \in (-\varepsilon, \varepsilon) \} = \mathbb{N}$

(ii) $\{ n \geq 1 \mid a_n \in (-\varepsilon, \varepsilon) \} \supseteq \{ 2k \mid k \geq 1 \}$

(iii) $\# \{ n \geq 1 \mid a_n \notin (-\varepsilon, \varepsilon) \} = \left\lfloor \frac{1}{\varepsilon} \right\rfloor$

(iv) $\# \{ n \geq 1 \mid a_n \notin (-\varepsilon, \varepsilon) \} = \left\lfloor \frac{1}{\varepsilon^4} \right\rfloor \sim \left(\frac{1}{\varepsilon} \right)^4$

2 (a) $a_n = \frac{(-1)^n}{n} \; \forall n \geq 1$

let $\varepsilon > 0$ be given.

observe $|a_n - 0| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n}$

$N = \left[\frac{1}{\varepsilon} \right] + 1 \qquad n \geq N \implies \frac{1}{n} < \varepsilon$

$\implies |a_n - 0| < \varepsilon$

As $\varepsilon > 0$ was arbitrary we have shown

$\forall \varepsilon > 0 \; \exists N \geq 1$ st $n \geq N \implies a_n \in (-\varepsilon, \varepsilon)$

let $N \geq 1$ be given

take $n_0 = 2N$ $\qquad a_{n_0} = \frac{(-1)^{n_0}}{n_0} = \frac{(-1)^{2N}}{2N} = \frac{1}{2N} > 0$

$n_1 = 2N+1$ $\qquad a_{n_1} = \frac{(-1)^{n_1}}{n_1} = \frac{(-1)^{2N+1}}{2N+1} = \frac{-1}{2N+1} < 0$

As $N \geq 1$. was arbitrary

$\forall N \geq 1 \quad \exists\, n_0 > N$ and $n_1 > N$ such that

$a_{n_0} > 0$ and $a_{n_1} < 0$

$\Rightarrow$ There are infinitely many elements of $\{a_n\}_{n \geq 1}$ that are positive

and " " " " " " negative